Contractions and Hamiltonian Line Graphs

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ABSTRACT

Using the contraction method, we find a best possible condition involving the minimum degree for a triangle-free graph to have a spanning eulerian subgraph.

I. INTRODUCTION

We follow Bondy and Murty [2] for basic terminology and notation, except for edge graphs and contractions. We shall use the term *line graph* for edge graph. For a graph G with a connected subgraph H, the *contraction* G/H is the multigraph obtained from G by replacing H by a vertex v_H, such that the number of edges in G/H joining any $v \in V(G - H)$ to v_H in G/H equals the number of edges joining v in G to $V(H)$. Note that multiple edges can arise in contractions. A graph is *eulerian* if it is connected and every vertex has even degree. (In particular, K_1, the graph with only one vertex and without edges, is eulerian.) An eulerian subgraph C of G is called a *spanning eulerian subgraph* of G if $V(C) = V(G)$ and is called a *dominating eulerian subgraph* of G if $E(G - V(C)) = \emptyset$. It is clear that any spanning eulerian subgraph is a dominating eulerian subgraph. For $v = V(G)$, we define the *neighborhood* $N(v)$ of v in G to be the set of vertices adjacent to v in G.

We present the following concept introduced by Catlin [3]:

A graph G is called *collapsible* if, for every subset S of $V(G)$ of even cardinality, there is a subgraph H of G such that

(i) $G - E(H)$ is connected, and
(ii) S is the set of vertices of odd degree in H.

The graph H will be called an S-*subgraph* of G.

Note that K_1 is collapsible, since $\emptyset$ is the only subset of $V(K_1)$ of even cardinality. All the complete graphs of order $n > 2$ are collapsible. In fact, Catlin

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[3] has shown that, if G is at most one edge short of having two edge-disjoint spanning trees, then G is collapsible or G has a cut-edge.

Catlin also proved the following theorem:

Theorem 1 (Catlin [3]). Let H_1 and H_2 be subgraphs of H such that $H_1 \cup H_2 = H$ and $H_1 \cap H_2 \neq \emptyset$. If H_1 and H_2 are collapsible, then H is collapsible.

One can define a relation on $V(G)$ as follows: for two vertices $x, y \in V(G)$, x is related to y if and only if $x, y \in V(H)$ for some collapsible subgraph H of G. Using Theorem 1, one can check that this is an equivalence relation on $V(G)$. The equivalence classes induce the maximal collapsible subgraphs of G. Let G_1 be obtained from G by contracting all maximal collapsible subgraphs of G. Catlin [3] has proved that G_1 has no nontrivial collapsible subgraphs. We shall call G_1 *the reduction of* G. Throughout this paper we shall use $d(v)$ and $d_1(v)$ to mean the degree of a vertex v in G and in G_1, respectively. A graph G is said to be *reduced* if it has no nontrivial collapsible subgraphs.

Define $V_4 = \{v \in V(G_1) : d_1(v) < 4\}$. Then we have

Theorem 2 (Catlin [3]). Let G be a graph and let G_1 be the reduction of G. Then

(i) G_1 is simple and K_3-free;
(ii) G has a spanning eulerian subgraph if and only if G_1 has a spanning eulerian subgraph;
(iii) If G_1 is nontrivial and 2-edge-connected, then $|V_4| \geq 4$, and if $|V_4| = 4$, then G_1 is eulerian.

We shall also use the following

Theorem 3 (Harary and Nash-Williams [4]). The line graph $L(G)$ of a simple graph G with at least three edges contains a hamiltonian cycle if and only if G has a dominating eulerian subgraph.

Bauer posed three problems in [1]:

HL1: Find "best possible" sufficient conditions on the vertex degrees of a simple graph G to ensure the existence of a hamiltonian cycle in its line graph $L(G)$.
HL2: Do HL1 for simple bipartite graphs.
HL3: Do HL1 for simple triangle-free graphs.

Recently Catlin settled HL1 with the following theorem:

Theorem 4 (Catlin [3]). If G is a 2-edge-connected simple graph on n vertices and $n \geq 20$, and if

$$\delta(G) > \frac{n}{5} - 1,$$

then $L(G)$ is hamiltonian.

Catlin also showed that this bound is best possible.

II. MAIN RESULTS

In this paper we settle HL2 and HL3. Our main theorem is given below.

Theorem 5. Let G be a 2-edge-connected triangle-free simple graph on $n \geq 31$ vertices. If

$$\delta(G) \geq n/10,$$

then exactly one of the following holds:

(i) G has a spanning eulerian subgraph;
(ii) $n = 10s$, for some integer $s \geq 4$, and G can be contracted to $G_1 = K_{2,3}$ in such a way that the preimage of each vertex of G_1 is either $K_{s,s}$ or $K_{s,s} - e$ for some edge e.

We start with the following lemma:

Lemma 6. A collapsible nontrivial triangle-free graph must have at least 6 vertices.

Proof. Denote by C_m the cycle of length m. Since a connected triangle-free graph with 2, 3, 4, or 5 vertices is either a tree, C_4, C_5, $K_{2,3}$, or C_4 with a pendant edge, and since none of these graphs is collapsible, a nontrivial triangle-free collapsible graph must have at least 6 vertices. ∎

Proof of Theorem 5. Let $H_1, H_2, \ldots, H_c$ denote the maximal collapsible subgraphs of G. Let G_1 be the reduction of G obtained from G by contracting the H_i's to distinct vertices $v_1, v_2, \ldots, v_c$. We re-index these vertices so that

$$d_1(v_1) \leq d_1(v_2) \leq \cdots \leq d_1(v_c).$$

If G_1 has a spanning eulerian subgraph, then by (ii) of Theorem 2, G has a spanning eulerian subgraph. Hence we may assume that G_1 has no spanning eulerian subgraphs. Since G is 2-edge-connected, it follows that G_1 is 2-edge-connected. Therefore $d_1(v_5) \leq 3$ by (iii) of Theorem 2. Let $i \in \{1,2,3,4,5\}$. Since $\delta(G) \geq 4$, the preimage H_i of v_i in G is nontrivial, and so $|V(H_i)| \geq 6$ by Lemma 6. Since $d_1(v_i) < 4$, we can find some $x_i \in V(H_i)$ such that $N(x_i) \subseteq V(H_i)$, where $N(x)$ denotes the neighborhood of x in G. Since

$$|N(x_i)| \geq \delta(G) \geq n/10 > 3 \geq d(v_i),$$

we can find $y_i \in N(x_i)$ such that $N(y_i) \subseteq V(H_i)$. Since G is triangle-free, $N(x_i) \cap N(y_i) = \emptyset$. Hence

$$n \geq \sum_{i=1}^{5} |V(H_i)| \geq \sum_{i=1}^{5} (|N(x_i)| + |N(y_i)|) \geq 10\delta(G) \geq n .$$

Therefore $c = 5$ and $|N(x_i)| = |N(y_i)| = \delta(G) = n/10$ and $V(H_i)$ is the union of the disjoint sets $N(x_i)$, $N(y_i)$ for $1 \leq i \leq 5$. Since G is triangle-free, this implies that H_i is a spanning subgraph of $K_{s,s}$ for $s = n/10$. Since $\delta(G) \geq n/10$ and $d_1(v_i) \leq 3$ edges of G have exactly one end in H_i, it follows that H_i is $K_{s,s}$ or $K_{s,s} - e$ for some $e \in E(K_{s,s})$.

Since G_1 is simple (by Theorem 2(i)), 2-edge-connected, and has no spanning eulerian subgraph and $c = 5$, it follows that $G_1 = K_{2,3}$. ∎

Remark. With a similar proof, we can show that if, $n = 30$ and if the inequality $\delta(G) \geq n/10$ is strict, then the conclusions of the theorem hold also.

The bound $n \geq 31$ is best possible in some sense. Let P_1, P_2, P_3 be three copies of the Petersen graph. Choose a vertex $v_1 \in V(P_1)$, two nonadjacent vertices $v_2, v_3 \in V(P_2)$, and two nonadjacent vertices $v_4, v_5 \in V(P_3)$. Let G be obtained from the union of P_1, P_2, and P_3 by adding four edges $v_1v_2, v_1v_3, v_1v_4, v_1v_5$. The resulting graph has 30 vertices and satisfies the hypothesis of Theorem 5, but it has no S-circuits, and it cannot be contracted to a $K_{2,3}$.

Corollary 7. Let G be a 2-edge-connected triangle-free simple graph on $n > 30$ vertices. If $\delta(G) > n/10$, then $L(G)$ is hamiltonian.

Proof. Apply Theorems 3 and 5. ∎

The graphs of (ii) of Theorem 5 also show that Corollary 7 is best possible.

Notice that some of the graphs of (ii) of Theorem 5 are bipartite graphs. It turns out that, for large n, Theorem 5 gives an answer to both HL2 and HL3.

We conclude by observing that Corollary 7 can be used to give, when $|V(G)| \geq 31$, alternative proofs of the following results that, however, were proved in [1] without this restriction on $|V(G)|$.

Corollary 8 (Bauer [1]). Let $G \subseteq K_{n,m}$ be bipartite, where $m \geq n \geq 2$. If $\delta(G) > m/2$, then $L(G)$ is hamiltonian.

Proof. By Corollary 7, it suffices to show that G is 2-edge-connected, which may be left to the reader. ∎

Corollary 9 (Bauer [1]). Let $G \subseteq K_{m,n}$ be bipartite and 2-connected. If $\delta(G) \geq (m + n + 5)/6$, then $L(G)$ is hamiltonian.

Proof. By Corollary 7. ∎

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References

[1] D. Bauer, On hamiltonian cycles in line graphs. Stevens Research Report, PAM No. 8501, Stevens Institute of Technology, Hoboken, NJ.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* American Elsevier, New York (1976).

[3] P. A. Catlin, A reduction method to find spanning eulerian subgraphs. Preprint.

[4] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* **9** (1965) 701–710.