UNIQUE GRAPH HOMOMORPHISMS ONTO ODD CYCLES

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ABSTRACT. A natural generalization of graph coloring is the graph homomorphisms. Let G and H be simple graphs. A map $\Phi: V(G) \to V(H)$ is called a *homomorphism* if Φ preserves adjacency. The set of all homomorphisms from G to H is denoted by $\mathrm{Hom}(G,H)$. If a homomorphism $\Phi \in \mathrm{Hom}(G,G)$ is bijective, Φ is called an *automorphism* of G. The set of all automorphisms of G forms a group $A(G)$. A graph G is *H-colorable* if $\mathrm{Hom}(G,H) \neq \emptyset$. G is *uniquely H-colorable* if $\mathrm{Hom}(G,H) \neq \emptyset$ and if for any Φ_1, $\Phi_2 \in \mathrm{Hom}(G,H)$, there exists an $\mathcal{A} \in A(H)$ such that $\mathcal{A}\Phi_1 = \Phi_2$. In this note, we proved best possible sufficient conditions involving $\delta(G)$ for a graph G to be uniquely Z_{2k+1}-colorable.

We shall use the notation of Bollobás [**1**]. Consider simple graphs G and H. A map $\Phi: V(G) \to V(H)$ is called a *homomorphism* if Φ preserves adjacency. The set of all homomorphisms from G to H is denoted by $\mathrm{Hom}(G,H)$. If a homomorphism $\Phi \in \mathrm{Hom}(G,G)$ is bijective, Φ is called an *automorphism* of G. The set of all automorphisms of G, denoted by $A(G)$, forms a group. Cycles of length m are denoted by C^m and occasionally by Z_m, for convenience [**3**].

A graph G is *H-colorable* if $\mathrm{Hom}(G,H) \neq \emptyset$. G is said to be *uniquely H-colorable* if $\mathrm{Hom}(G,H) \neq \emptyset$ and if for any $\Phi_1, \Phi_2 \in \mathrm{Hom}(G,H)$, there exists an $\mathcal{A} \in A(H)$ such that $\mathcal{A}\Phi_1 = \Phi_2$ [**3**].

In 1976, Bollobás proved in [**2**] that if G is of order n and $\mathrm{Hom}(G,K^k) \neq \emptyset$ $\delta > n(3k-5)/(3k-2)$, then G is uniquely K^k-colorable. Bollobás also proved that if there exists $\Theta \in \mathrm{Hom}(G,K^k)$ such that $G[\Theta^{-1}(i) \cup \Theta^{-1}(j)]$ is connected for all $i, j \in V(K^k)$, then the inequality can be improved to $\delta > n(k-2)/(k-1)$ to guarantee G to be uniquely K^k-colorable. Bollobás gave classes of graphs to show that these conditions are best possible. Catlin, in his papers [**3**] and [**4**], used graph homomorphisms as a generalization of the coloring of graphs. In this note we shall give, analogous to Bollobás' first theorem, best possible sufficient conditions involving $\delta(G)$ for a graph to be uniquely C^{2k+1}-colorable.

In the following we always assume $k > 1$ since the case $k = 1$ has been discussed in Bollobás' paper [**2**].

Let $k > 1$ and $h > 1$ be integers. As usual, Z_{2k+1} is an odd cycle of length $2k+1$ and with its vertices marked with integers modulo

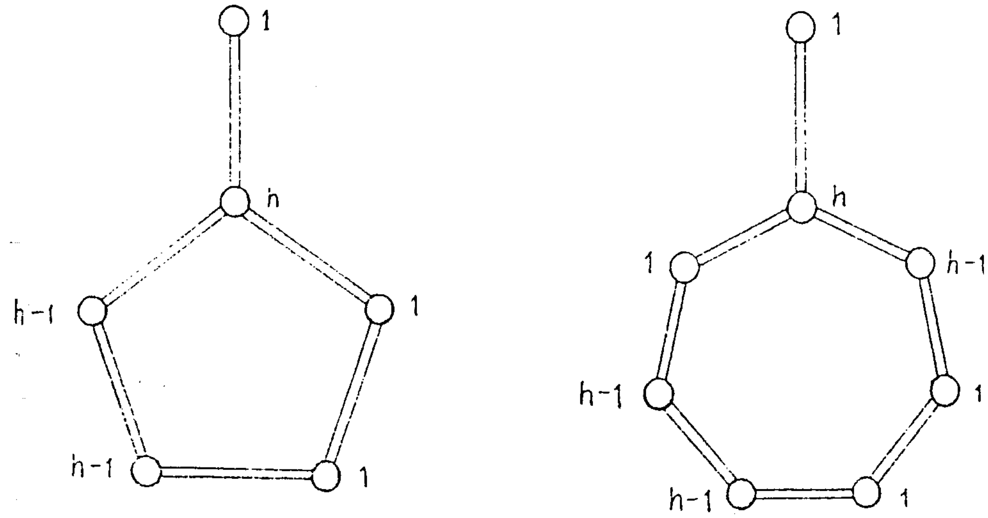
UTILITAS MATHEMATICA 31 (1987), 199-208.

$2k+1$. Let H be the graph obtained by adding to Z_{2k+1} a new vertex marked with -1 which is adjacent only to 0. Let $G(k,h)$ be the graph obtained from H by replacing each vertex i of H, where either $i=-1$ or $i \in Z_{2k+1}$, by a set S_i called a *cloud* of n_i vertices as follows:

If k is even, then $n_{-1}=1$, $n_0=h$, $n_{4j+1}=n_{4j+2}=1$, $n_{4j+3}=n_{4j+4}=h-1$, where $j=0,1,2,3,\ldots,(k/2-1)$.

If k is odd, then $n_{-1}=1$, $n_0=h$, $n_{2k}=n_{4j+2}=n_{4j+3}=1$, $n_1=n_{4j+5}=n_{4j+4}=h-1$, where $j=0,1,2,\ldots,((k-1)/2-1)$.

Thus two vertices of $G(k,h)$ are joined if and only if they belong to different sets S_i that were adjacent in H.



(a) $G(2,h)$ (b) $G(3,h)$

Figure 1

In Figure 1, each circle represents a cloud. The number beside each circle is the number of vertices of the cloud. The lines represent all possible edges between the clouds.

Then $G(k,h)$ is a graph of order $n=(k+1)h+1$ and $\delta(G(k,h))=h$. It is obvious that

$$\delta(G(k,h))=(n-1)/(k+1) \quad \text{and} \quad \lim_{n\to\infty} h/n = 1/(k+1).$$

Also, $G(k,h)$ is not uniquely Z_{2k+1}-colorable. This example shows that the theorem below is best possible.

LEMMA 1.

(i) *If there exists a $\Theta \in Hom(G, Z_{2k+1})$ such that $\Theta(G)=Z_{2k+1}$, then*

$$\delta(G) \le 2n/(2k+1).$$

(ii) *If, in addition,* $\delta(G) > n/(k+1)$, *then* $\Theta_M < n/(k+1)$, *where*

$$\Theta_M = \max_{i \in Z_{2k+1}} \{|\Theta^{-1}(i)|\}.$$

(iii) *If, in addition to (i),* $\delta(G) \geq n/(k+1)$, *then* $\Theta_M \leq n/(k+1)$.

PROOF:

(i) Let $n_i = |\Theta^{-1}(i)|$, $i \in Z_{2k+1}$. Clearly we have $n_i + n_{i+2} \geq \delta(G)$, $\forall i \in Z_{2k+1}$. Adding all these inequalities, we have

$$2n = \sum_{i \in Z_{2k+1}} (n_i + n_{i+2}) \geq (2k+1)\delta(G).$$

Hence, $\delta(G) \leq (2n)/(2k+1)$.

(ii) Suppose, to the contrary, $\Theta_M \geq n/(k+1)$. We may assume that $n_0 \geq n/(k+1)$. Note that $\delta(G) > n/(k+1)$ and $\Theta(G) = Z_{2k+1}$ imply that:

$$n_i + n_{i+2} > n/(k+1), \quad \forall i \in Z_{2k+1}.$$

Case 1. Suppose k is even. Let $s = k/2$ and consider the following inequalities:

$$\begin{aligned} &n_1 + n_3 > n/(k+1); \\ &n_2 + n_4 > n/(k+1); \\ &\cdots \\ &n_{4s-3} + n_{4s-1} > n/(k+1); \\ &n_{4s-2} + n_{4s} > n/(k+1). \end{aligned}$$

We have k of them in all. Hence

$$n = \sum_{i \in Z_{2k+1}} n_i > n/(k+1) + (kn)/(k+1) = n,$$

a contradiction.

Case 2. Suppose k is odd. We consider another set of inequalities:

$$\begin{aligned} &n_1 + n_{2k} > n/(k+1); \\ &n_2 + n_4 > n/(k+1); \\ &n_3 + n_5 > n/(k+1); \\ &\cdots \\ &n_{2k-4} + n_{2k-2} > n/(k+1); \\ &n_{2k-3} + n_{2k-1} > n/(k+1). \end{aligned}$$

Again we have a contradiction.

(iii) The proof is similar to that of (ii).

REMARK: Using the same idea in the proof of (ii) and (iii), we can get the following generalization:

$$\Theta_M \leq n - k\delta(G),$$

provided that $\Theta(G) = Z_{2k+1}$.

COROLLARY. *Suppose that there exists a* $\Theta \in Hom(G, Z_{2k+1})$ *such that* $\Theta(G) = Z_{2k+1}$. *If*

(i) $\delta(G) > n/(k+1)$, *or*

(ii) $\delta(G) \geq n/(k+1)$ *and* $|\Theta^{-1}(j)| < n/(k+1)$ *for all* $j \in Z_{2k+1}$, *then,* $\forall x \in \Theta^{-1}(i)$, *we have* $\Gamma(x) \cap \Theta^{-1}(i+1) \neq \emptyset$ *and* $\Gamma(x) \cap \Theta^{-1}(i-1) \neq \emptyset$ *for all* $i \in Z_{2k+1}$.

PROOF: Since

$$|\Gamma(x)| \geq \delta(G) > \Theta_M,$$

by Lemma 1, the corollary follows.

LEMMA 2. *Suppose that there exists a* $\Theta \in Hom(G, Z_{2k+1})$ *such that* $\Theta(G) = Z_{2k+1}$. *If:*

(i) $\delta(G) > n/(k+1)$, *or*

(ii) $\delta(G) \geq n/(k+1)$ *and* $|\Theta^{-1}(j)| < n/(k+1)$ *for all* $j \in Z_{2k+1}$, *then every vertex of* G *lies in a* C^{2k+1}*-subgraph of* G.

Note that the conclusion of this lemma assures the existence of C^{2k+1}*-subgraphs of* G.

PROOF: Before we start the proof, we introduce some notations. For subsets S_1, S_2 of $V(G)$, we denote:

$$(S_1,\, S_2) = \{v \in S_2 \colon wv \in E(G) \quad \text{for some} \quad w \in S_1\}.$$

If $S_1 = \{v\}$, we then write (v, S_2) for $(S_1,\, S_2)$.

Pick an arbitrary vertex $v_0 \in \Theta^{-1}(0)$. Let $V_1 = (v_0, \Theta^{-1}(1))$ and $V_i = (V_{i-1},\, \Theta^{-1}(i))$, $i = 2, 3, \ldots 2k-1$. Let $V_{2k} = (v_0,\, \Theta^{-1}(2k))$, $W_{2k} = (V_{2k-1}, \Theta^{-1}(2k))$ and $W_{2k-1} = (V_{2k},\, \Theta^{-1}(2k-1))$. The previous corollary says that all these sets are nonempty.

It is clear that if $W_{2k-1} \cap V_{2k-1} \neq \emptyset$ or $W_{2k} \cap V_{2k} \neq \emptyset$, we have a C^{2k+1} contained in G and containing v_0. Hence it suffices to show

$$W_{2k-1} \cap V_{2k-1} \neq \emptyset \quad \text{or} \quad W_{2k} \cap V_{2k} \neq \emptyset.$$

Suppose both sets $W_{2k-1} \cap V_{2k-1}$ and $W_{2k} \cap V_{2k}$ are empty. First we assume that (i) holds.

Case 1. Suppose k is even. Let $k = 2s$ for some natural number Consider the degrees of the vertices in the sets V_{2k}, $\{v_0\}$, $V_1, V_2, \ldots$ We have

$$|W_{2k-1}| + |\Theta^{-1}(0)| > n/(k+1);$$
$$|V_{2k}| + |V_1| > n/(k+1);$$
$$|W_{2k}| + |\Theta^{-1}(2k-2)| > n/(k+1);$$
$$|\Theta^{-1}(2k-3)| + |V_{2k-1}| > n/(k+1);$$

together with the following $k-2$ inequalities:

$$|\Theta^{-1}(4j+1)| + |V_{4j+3}| > n/(k+1);$$
$$|\Theta^{-1}(4j+2)| + |V_{4j+4}| > n/(k+1);$$

where $j = 0, 1, \ldots s-2$.

Adding all these inequalities and noting that the emptiness of W_{2k} ∩ V_{2k} and $W_{2k-1} \cap V_{2k-1}$ implies

$$|V_{2k}| + |W_{2k}| \leq |\Theta^{-1}(2k)|,$$
$$|V_{2k-1}| + |W_{2k-1}| \leq |\Theta^{-1}(2k-1)|,$$

we get

$$|\Theta^{-1}(1)| + \sum_{i \in Z_{2k+1}} |\Theta^{-1}(i)| > n + n/(k+1)$$

contrary to the results of Lemma 1.

Case 2. Suppose k is odd. We write $k = 2s+1$. Considering th degrees as before, we have

$$|W_{2k-1}| + |\Theta^{-1}(0)| > n/(k+1);$$
$$|W_{2k}| + |\Theta^{-1}(2k-2)| > n/(k+1);$$
$$|V_{2k-1}| + |\Theta^{-1}(2k-3)| > n/(k+1);$$
$$|V_{2k}| + |V_1| > n/(k+1);$$
$$|\Theta^{-1}(0)| + |V_2| > n/(k+1);$$

together with the following $k-3$ inequalities:

$$|\Theta^{-1}(4j+3)| + |V_{4j+5}| > n/(k+1);$$
$$|\Theta^{-1}(4j+4)| + |V_{4j+6}| > n/(k+1);$$

where $j = 0, 1, \ldots, s-2$.

Adding these inequalities as in Case 1, we get

$$|\Theta^{-1}(0)| + \sum_{i \in Z_{2k+1}} |\Theta^{-1}(i)| > n + n/(k+1),$$

which is a contradiction.

The proof when (ii) holds is similar. Hence the lemma.

Now we are ready to prove the main theorem.

THEOREM. *If, for $k > 1$,*

(1) $\delta(G) \geq n/(k+1)$, and

(2) there exists a $\Theta \in Hom(G, Z_{2k+1})$ such that $\Theta(G) = Z_{2k+1}$, then G is uniquely Z_{2k+1}-colorable.

PROOF: For each graph G having at least one C^{2k+1} as its subgraph, we define a new graph, denoted by $C^{2k+1}(G)$, whose vertex set is the set of all C^{2k+1}-subgraphs of G, where two vertices of $C^{2k+1}(G)$ are adjacent if and only if the corresponding C^{2k+1} subgraphs of G have at least one edge in common.

A homomorphism from G to Z_{2k+1} can be regarded as an orientation of the C^{2k+1}-subgraphs of G. Hence it is clear that if in a graph G, every vertex is in a C^{2k+1}-subgraph of G and $C^{2k+1}(G)$ is connected, then G is uniquely Z_{2k+1}-colorable.

Based on the above idea, we begin our proof. We shall divide the proof into a few cases.

Case 1. Assume that $\delta > n/(k+1)$. In this case, Lemma 2 says that every vertex of G lies in a C^{2k+1}-subgraph of G. We need only show $C^{2k+1}(G)$ is connected.

Fix a C^{2k+1}-subgraph of G. Let $v_0, v_1, \ldots v_{2k}, v_0$ be the vertices of this C^{2k+1}, such that $\Theta(v_i) = i$, for all $i \in Z_{2k+1}$. Let

$$N(C^{2k+1}) = \bigcup_{i \in Z_{2k+1}} \Gamma(v_{i+1}) \cap \Gamma(v_{i-1}).$$

Note that

$$\Gamma(v_i) = \left[\Gamma(v_i) \cap \Theta^{-1}(i+1)\right] \cup \left[\Gamma(v_i) \cap \Theta^{-1}(i-1)\right] \quad \text{for all } \ i \in Z_{2k+1}.$$

We have

$$\begin{aligned}
|N(C^{2k+1})| &= \sum_{i \in Z_{2k+1}} |\Gamma(v_{i+1}) \cap \Gamma(v_{i-1})| \\
&= \sum_{i \in Z_{2k+1}} \left| \left[\Gamma(v_{i+1}) \cap \Theta^{-1}(i)\right] \cap \left[\Gamma(v_{i-1}) \cap \Theta^{-1}(i)\right] \right| \\
&= \sum_{i \in Z_{2k+1}} \{|\Gamma(v_{i+1}) \cap \Theta^{-1}(i)| + |\Gamma(v_{i-1}) \cap \Theta^{-1}(i)| \\
&\quad -|\Theta^{-1}(i) \cap [\Gamma(v_{i+1}) \cup \Gamma(v_{i-1})]|\} \\
&\geq \sum_{i \in Z_{2k+1}} \{|\Gamma(v_{i+1}) \cap \Theta^{-1}(i)| + |\Gamma(v_{i-1}) \cap \Theta^{-1}(i)| - |\Theta^{-1}(i)|\} \\
&= \sum_{i \in Z_{2k+1}} \deg_G(v_i) - n \\
&> (2k+1)n/(k+1) - n = kn/(k+1).
\end{aligned}$$

Hence for any two C^{2k+1} subgraphs C_1^{2k+1}, C_2^{2k+1} of G,

$$|N(C_1^{2k+1}) \cap N(C_2^{2k+1})| \geq |N(C_1^{2k+1})| + |N(C_2^{2k+1})| - n > (k-1)n/(k+1).$$

By Lemma 1, $\Theta_M < n/(k+1)$, we must have at least k i's such that

$$N(C_1^{2k+1}) \cap N(C_2^{2k+1}) \cap \Theta^{-1}(i) \neq \emptyset. \tag{3}$$

In the following we also use C_1^{2k+1}, C_2^{2k+1} to denote the corresponding vertices in $C^{2k+1}(G)$. We shall show that there exists a path in $C^{2k+1}(G)$ connecting C_1^{2k+1} and C_2^{2k+1}. First, let us assume that as C^{2k+1}-subgraphs of G, C_1^{2k+1} and C_2^{2k+1} have one vertex in common. We may assume that

$$C_1^{2k+1} = x_0,\ x_1, \ldots x_{2k},$$
$$C_2^{2k+1} = y_0,\ y_1, \ldots y_{2k},$$

such that $x_0 = y_0$ and $\Theta(x_j) = \Theta(y_j) = j$ for all $j \in Z_{2k+1}$. Let i satisfy (3).

Subcase 1. Suppose $i = 1$. Let $z \in N(C_1^{2k+1}) \cap N(C_2^{2k+1}) \cap \Theta^{-1}(1)$. And let

$$C_3^{2k+1} = x_0,\ z,\ x_2, \ldots, x_{2k}$$
$$C_4^{2k+1} = y_0,\ z,\ y_2, \ldots, y_{2k}.$$

It is clear that C_1^{2k+1}, C_3^{2k+1}, C_4^{2k+1}, C_2^{2k+1} is a path in $C^{2k+1}(G)$. The proof for $i = 2k$ is similar.

Subcase 2. Suppose $i \neq 0$, $i \neq 1$, and $i \neq 2k$. Let

$$w \in N(C_1^{2k+1}) \cap N(C_2^{2k+1}) \cap \Theta^{-1}(i),$$

and let

$$C_5^{2k+1} = y_0, \ldots y_{i-1},\ w,\ x_{i+1}, \ldots x_{2k}.$$

Clearly C_1^{2k+1}, C_5^{2k+1}, C_2^{2k+1} is a path in $C^{2k+1}(G)$.

Hence, if C_1^{2k+1} and C_2^{2k+1} have one vertex in common, then C_1^{2k+1} and C_2^{2k+1} are connected by a path in $C^{2k+1}(G)$.

Now consider Case 1 when $V(C_1^{2k+1}) \cap V(C_2^{2k+1}) = \emptyset$. We adopt the same notations for C_1^{2k+1}, C_2^{2k+1}, but $x_0 \neq y_0$ now. By (3), we may assume that

$$\Theta^{-1}(0) \cap N(C_1^{2k+1}) \cap N(C_2^{2k+1}) \neq \emptyset.$$

Let $u \in \Theta^{-1}(0) \cap N(C_1^{2k+1}) \cap N(C_2^{2k+1})$ and let

$$C_6^{2k+1} = u,\ x_1, \ldots, x_{2k}$$
$$C_7^{2k+1} = u,\ y_1, \ldots, y_{2k}.$$

By what we have just proved, C_6^{2k+1} and C_7^{2k+1} are connected by a path in $C^{2k+1}(G)$ since they have a common vertex u. Clearly C_1^{2k+1}, C_6^{2k+1} and C_2^{2k+1}, C_7^{2k+1} are paths of $C^{2k+1}(G)$. Hence C_1^{2k+1} and C_2^{2k+1} are connected by a path in $C^{2k+1}(G)$. This means $C^{2k+1}(G)$ is connected.

Case 2. Assume $\delta(G) = n/(k+1)$ and $|\Theta^{-1}(i)| < n/(k+1)$ for all $i \in Z_{2k+1}$.

The proof of this case is similar to that of Case 1. Lemma 2 says that in this case every vertex of G lies in a C^{2k+1} subgraph of G. So we shall show that $C^{2k+1}(G)$ is connected.

Changing $\deg_G(v_i) > n/(k+1)$ into $\deg_G(v_i) \geq n/(k+1)$ in the previous estimation of $|N(C^{2k+1})|$, we get this time

$$|N(C^{2k+1})| \geq kn/(k+1).$$

Hence, for any C_1^{2k+1}, $C_2^{2k+1} \in V(C^{2k+1}(G))$, we have

$$|N(C_1^{2k+1}) \cap N(C_2^{2k+1})| \geq n(k-1)/(k+1).$$

The assumption that $|\Theta^{-1}(i)| < n/(k+1)$ for all $i \in Z_{2k+1}$ implies that we must have at least k i's such that (3) holds. This implies that $C^{2k+1}(G)$ is connected, as we proved before.

Case 3. Assume $\delta(G) = n/(k+1)$, and $|\Theta^{-1}(i)|$ equals $n/(k+1)$, for some $i \in Z_{2k+1}$.

We may assume $|\Theta^{-1}(0)| = n/(k+1)$. Let $h = n/(k+1)$, and $n_i = |\Theta^{-1}(i)|$ for all $i \in Z_{2k+1}$.

Subcase 1. k is even. Let $k = 2s$ for some positive integer s. We have the following inequalities:

$$\begin{aligned} n_{4j+3} &\geq h - n_{4j+1}, \\ n_{4j+4} &\geq h - n_{4j+2}, \end{aligned} \tag{4}$$

$j = 0, 1, 2, \ldots, s-1$, since $\delta(G) = h$.

We have

$$n = \sum_{i \in Z_{2k+1}} n_i \geq h + \sum_{j=0}^{s-1} (n_{4j+1} + n_{4j+2} + h - n_{4j+1} + h - n_{4j+2}).$$

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Received June 10, 1985; revised March 13, 1986