Part III

SUBGRAPHS

7. Subgraphs of graphs, II

In [6] we gave a sufficient condition for H to be a subgraph of G by showing that for any positive integer d there is a constant $c_d < d$ such that $\Delta(H) \leq d$ and $\delta(G) \geq c_d p$ imply that H is a subgraph of G. We obtained this from a result on bipartite graphs that is analogous to Theorem 7.1, and is in a certain sense best possible. In a footnote in [6] we announced having improved c_d to the value given by Theorem 7.1 below. Like Theorem 7.1, our result in [6] on bipartite graphs may be obtained using a generalization of the concept of alternating paths, which is used extensively in studying matchings. In the special case when $\Delta(H) = 1$, the proof of Theorem 7.1 reduces to an argument involving an alternating path of length 4.

N. Sauer and J. Spencer [14] have independently obtained Theorem 7.1. This was announced in [13]. Erdős and Stone [9] gave a sufficient condition of a different nature for H to be a subgraph of G. Bollobás and Eldridge [3] and Sauer and Spencer [14] have considered the problem of giving sufficient conditions, based on the number of edges of H and G^c, for H to be a subgraph of G.

After proving the main result of section 7, we give some special cases, and indicate what would be best possible. Our main result of this section is

Theorem 7.1 If G and H are graphs on p vertices such that

$$(7.1) \qquad 2\Delta(G^c)\Delta(H) \leq p-1$$

then H is a subgraph of G.

Proof: Throughout the proof, the letter w will be used to denote vertices of H (i.e., $w' \in V(H)$), and the letters x and v will be used for vertices of G. Given a graph G, suppose that H is an edge-minimal graph that is not a subgraph of G, but suppose that H and G satisfy (7.1). By edge-minimality, we can pick any edge, say $e \in V(H)$, fixed throughout the proof, so that $H - e$ is a subgraph of G. Let

$$\pi\colon V(H) \longrightarrow V(G)$$

be an embedding of $H - e$ into G. Let w, w' be the ends of e in H. We shall alter π by transposing $\pi(w)$ with another vertex of G so that the resulting embedding of $H - e$ also maps e to an edge of G. This is a contradiction.

Define

$$M(v) = \{v'' \in V(G): \{\pi^{-1}(v), \pi^{-1}(v'')\} \in E(H-e)\}.$$

A successor of v is any vertex $v_1 \in V(G)$ such that for each $v'' \in M(v)$, v_1 is either equal or adjacent in G to v". Denote by S(v) the set of all successors of v. We say that v is a predecessor of v_1 if v_1 is a successor of v. Denote by $P(v_1)$ the set of all predecessors of v_1.

Let $v = \pi(w)$. Note that if $v_1 \in S(v) \cap P(v)$ and if $v_1 \neq v$, then $(v\ v_1)\pi$ embeds $H-e$ into G.

We estimate $|S(v)|$ and $|P(v)|$ by deriving upper bounds for $|V(G) - S(v)|$ and $|V(G) - P(v)|$. A vertex $x \in V(G)$ is outside S(v) if x is adjacent in G^c to a vertex x' of M(v). For any given $x' \in M(v)$, there are $\Delta(G^c)$ choices of x adjacent to x' in G^c. Since $\deg_{H-e}(w) \leq \Delta(H) - 1$, we must have $|M(v)| \leq \Delta(H) - 1$ choices of x'. Hence, at most $\Delta(G^c)(\Delta(H) - 1)$ vertices x are not in S(v). If $x \notin P(v)$, then there is an $x' \in M(x)$ such that x' is adjacent in G^c to v. There are at most $\Delta(G^c)$ choices of x' adjacent to v in G^c, and one of them is $\pi(w')$, since π does not embed e into G. Each x' lies in at most $\Delta(H)$ sets M(x), where $x \in V(G)$, with strict inequality when $\pi^{-1}(x') = w'$, whence at most $\Delta(G^c)\Delta(H) - 1$ vertices x of V(G) are not in P(v).

Therefore,

$$\begin{aligned} |P(v) \cap S(v)| &\geq |V(G)| - |V(G) - P(v)| - |V(G) - S(v)| \\ &\geq p - (\Delta(G^c)\Delta(H) - 1) \\ &\quad - \Delta(G^c)(\Delta(H) - 1) \\ &= p - 2\Delta(H)\Delta(G^c) + \Delta(G^c) + 1 \\ &\geq 2 + \Delta(G^c), \end{aligned}$$

by (7.1). At most $1 + \Delta(G^c)$ vertices are not adjacent in G to $\pi(w')$. Therefore, there is a $v_1 \in P(v) \cap S(v)$ that is adjacent to $\pi(w')$ in G. Thus, $(v\ v_1)\pi$ is an embedding of H into G. This proves the theorem.

<u>Conjecture</u> If G and H are graphs on p vertices satisfying

$$(\Delta(H) + 1)(\Delta(G^c) + 1) \leq p + 1,$$

then H is a subgraph of G.

We give examples to show that the conjecture, if true, would be best possible. Let H be a graph on p vertices, and let d be an integer such that

$$p \equiv -2 \pmod{d + 1}.$$

Then H is said to be in class $C_1(d)$ if $\Delta(H) = d$ and if H has $\frac{p+2}{d+1} - 1$ components isomorphic to K_{d+1}; H is in class $C_2(d)$ if d is odd, if H has one component isomorphic to $K_{d,d}$, and if all $\frac{p+2}{d+1} - 2$ other components are isomorphic to K_{d+1}. Thus, for any odd d, there is a unique graph in $C_2(d)$, and for d even, $C_2(d)$ is empty. We also denote these classes by C_1 and C_2 (Figures 1,2,3).

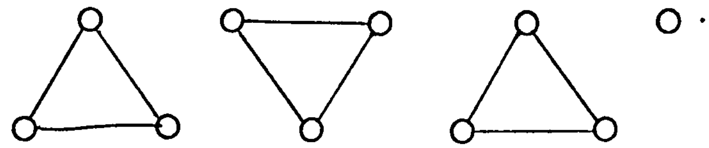

Figure 1: The graph in $C_1(2)$ with $p = 10$.

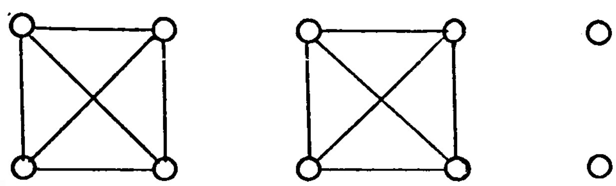

Figure 2: A graph in $C_1(3)$ with $p = 10$.

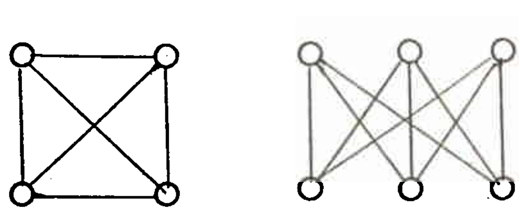

Figure 3: The graph in $C_2(3)$ with $p = 10$.

For any integers d,d' satisfying

$$(d+1)(d'+1) = p+2,$$

if $H \in C_1(d) \cup C_2(d)$ and if $G^c \in C_1(d') \cup C_2(d')$, then it is easily verified that H is not a subgraph of G, unless $H \in C_2(d)$ and $G^c \in C_2(d')$. Thus, the conjecture, if true, is best possible.

It is routine to show that if (7.1) of Theorem 7.1 is improved to

$$2\Delta(H)\Delta(G^c) \leq p,$$

then either H is a subgraph of G, or one of H or G^c, say H, is regular of degree 1, and the other graph lies in $C_1(p/2)$ or $C_2(p/2)$. The proof, however, becomes much longer than the proof of Theorem 7.1.

We have recently been able to show that there is a function f(p), on the order of $p^{1/3}$, such that if G and H are graphs on p vertices with $\Delta(H) = 2$ and $\Delta(G^c) \leq \frac{p}{3} - f(p)$, then H is a subgraph of G. The coefficient $\frac{1}{3}$ is best possible by above examples. A proof is in section 10. In the special case where H has $[\frac{p}{3}]$ components isomorphic to K_3 and G satisfies

$$\Delta(G^c) \leq \frac{p-1}{3},$$

either H is a subgraph of G, or equality holds and $G^c \in C_1(\frac{p-1}{3}) \cup C_2(\frac{p-1}{3})$. This characterizes the extremal graphs of a theorem of Corrádi and Hajnal [7]. We prove this in section 9.

There is a more general special case for which the conjecture has been proved. If

$$b = \frac{p+1}{\Delta(H)+1} - 1$$

is an integer, then this value of b in the following theorem of Hajnal and Szemeredi [10] gives the inequality of the conjecture. The case $\Delta(H) = 2$ is the theorem of Corrádi and Hajnal.

<u>Theorem 7.2</u> Let H be a graph with b components isomorphic to $K_{\Delta(H)+1}$, and with all other components isomorphic to $K_{\Delta(H)}$. If

$$\Delta(G^c) \leq \frac{p-b}{\Delta(H)} - 1,$$

then H is a subgraph of G.

Theorem 7.2 can be readily derived from the special case with $b = \frac{p+1}{\Delta(H)+1} - 1$. In this case $H \in C_1$. Hajnal and Szemeredi gave $G^c \in C_1$ to show that their result is best possible. The example $G^c \in C_2$ is new.

Given a graph H, the conjecture is not necessarily best possible for that particular graph. For instance, consider the following theorem of Bondy [4]:

<u>Theorem 7.3</u> If H is a graph consisting of one polygonal component of girth $g \leq p$ and of $p - g$ components K_1, and if G satisfies $\Delta(G^c) \leq \frac{1}{2}p - 1$ and also has p vertices, then either H is a subgraph of G, or g is odd, p is even and G is isomorphic to $K_{\frac{1}{2}p,\frac{1}{2}p}$.

Let $n = [\frac{p}{2}]$. If G^c contains a component isomorphic to K_{n+1} and $g = p$, or if G^c has a component isomorphic to $K_{n,n}$ and $g > p - n$, then H, of Theorem 7.3, is not a subgraph of G. These examples are similar to the classes C_1 and C_2 which make Theorem 7.1 best possible for $\Delta(H) = 1$, and which make the conjecture best possible.

Next, we give a general construction of examples of graphs G such that a given graph H is not a subgraph of G. This construction generalizes class C_1 given above.

Let r(H) denote the minimum number of vertices whose removal from H is necessary to lower the chromatic number $\chi(H)$. Clearly,

$$r(H) \leq \frac{p}{\chi(H)} .$$

Theorem 7.4 Let H be a graph on p vertices. There is a graph G on p vertices with

$$\Delta(G^c) = \left[\frac{p - r(H)}{\chi(H) - 1}\right]$$

such that H is not a subgraph of G.

Proof: Let $n = \chi(H) - 1$. Partition the p vertices of a set X into sets $X_0, X_1, \ldots, X_n$, where

$$|X_0| = r(H) - 1,$$

where X_0 is empty if $r(H) = 1$, where

$$|X_1| \leq |X_2| \leq \ldots \leq |X_n|,$$

and $|X_n| - |X_1| \leq 1$. Let a graph G be defined on X so that G is the complete $(n+1)$-partite graph with $(n+1)$-partition $X_0, X_1, \ldots, X_n$. Since

$$|X_0| = r(H) - 1 \le \frac{p}{\chi(H)} - 1,$$

and since $\frac{p}{\chi(H)}$ is the average of $|X_i|$, $i = 0,1,\ldots,n$, we must have $|X_n| > |X_0|$. Thus, $G^c[X_n]$ is a clique, and letting braces denote the least integer function, we have

$$\begin{aligned}\Delta(G^c) &= |X_n| - 1\\ &= \{(p - |X_0|)/n\} - 1\\ &= [(p - |X_0| + n - 1)/n] - 1\\ &= [(p - |X_0| - 1)/n]\\ &= [(p - r(H))/(\chi(H) - 1)],\end{aligned}$$

which is the condition of the theorem.

Suppose that π embeds H into G. Let H' denote the subgraph of H induced by the preimage of $V(G) - X_0$. Since H' contains $|X_0| = r(H) - 1$ fewer vertices than H, by the definition of r,

$$\chi(H') = \chi(H).$$

But for each $i = 1,2,\ldots,n$, no two vertices of X_i are adjacent in G, and so the embedding

$$\pi|_{H'}: V(H') \longrightarrow V(G) - X_0$$

is a $(\chi(H) - 1)$-coloring of H', a contradiction. Hence, π does not exist, and the theorem is proved.

A B_h-component is defined in section 2.

<u>Corollary 7.5</u> Let H be a graph on p vertices of maximum degree $\Delta(H)$ with $b > 0$ B_h-components. Then there is a graph G with $\Delta(G^c) = [(p - b)/\Delta(H)]$, such that H is a subgraph of G.

Proof: By Theorem 2.1, due to Brooks, we have $\chi(H) = \Delta(H)+1$, $r(H)=b$ in Theorem 7.4.

Note that Theorem 7.4 and Corollary 7.5 contain, as special cases, some of the aforementioned examples showing that certain results are best possible.

Consider Bondy's Theorem. If p and g are odd, then $b=1$ and Corollary 7.5 shows Theorem 7.3 to be best possible. If g is even and equal to p, then $r(H) = \frac{1}{2}p$, and Theorem 7.4 shows that $\Delta(G^c) \leq \frac{1}{2}p$ is not sufficient to ensure that H is a subgraph of G.

In the case of Hajnal and Szemeredi's Theorem, Corollary 7.5 ensures that it is best possible for any value of $\Delta(H)$ and any $b \geq 1$.

Let H be a graph satisfying the conditions of Corollary 7.5. We know of no graph on p vertices with

$$\Delta(G^c) \leq \frac{p-b}{\Delta(H)} - 1,$$

even when $b=0$, such that H is not a subgraph of G. It would be interesting to know whether such graphs exist.

8. Subgraphs of maximum degree 2: a short proof.

We give in this section an improvement of Theorem 7.1 for the case $\Delta(H) = 2$ that has a short proof, but is not best possible. In section 10, we prove a stronger result, which is best possible in a certain sense.

As in section 7, the graph H will be embedded in G, and the letters y and w will be used to denote the vertices of H, while x and v will be used for vertices of G. In this section, let

$$M(v) = \{v'' \in V(G) : \{\pi^{-1}(v), \pi^{-1}(v'')\} \in E(H)\},$$

where

$$\pi : V(H) \longrightarrow V(G)$$

is a fixed bijection.

In this section we used a slightly different definition of successors and predecessors. A vertex $x' \in V(G)$ is a <u>successor</u> of $x \in V(G)$ if x' is adjacent in G to every vertex of $M(x)$. (In section 7, we permitted x' to be a successor of x if x' were adjacent or equal to every vertex of $M(x)$.) The vertex x is a <u>predecessor</u> of x' if x' is a successor of x.

<u>Theorem 8.1</u> Let G and H be graphs on p vertices, with $\Delta(H) = 2$. If

$$(8.1) \qquad \Delta(G^c) \leq \frac{2p-11}{7},$$

then H is a subgraph of G.

<u>Proof</u>: Let H be an edge-minimal graph for which the theorem is false. Then there is a graph G satisfying (8.1) such that H is not a subgraph of G, but such that for any edge $e \in E(H)$, $H - e$ is a subgraph of G.

First, we show that if any vertex $y_1 \in V(H)$ has degree 1 in H, then we are done. Let $e = \{y_0, y_1\}$ be the edge incident with y_1, and let

$$\pi: V(H) \longrightarrow V(G)$$

be an embedding of $H - e$ into G. Since

$$\Delta(G^c) < \frac{2p}{7},$$

$\pi(y_1)$ has at least $\frac{3p}{7}$ predecessors in G, and (8.1) guarantees that among them lie successors of $\pi(y_1)$ (i.e., vertices of G adjacent to $\pi(y_0)$). Let x be such a vertex. Then $(x\ \pi(y_1))\pi$ is an embedding of H into G.

Therefore, assume that each vertex of H has degree either 0 or 2. Thus, all components of H are either isolated vertices or polygons.

Let the edges of polygons of H be directed so that each vertex has one incoming edge and one outgoing edge.

Given a vertex y_0 in H, we shall denote by y_1, y_2, y_3 the next three successive vertices on the directed path in H starting at y_0. On a triangle, $y_0 = y_3$. For the mapping

$$\pi: \ V(H) \longrightarrow V(G),$$

we shall simplify notation by writing $\pi(y_i) = x_i$ for $i = 0,1,2,3$.

By the minimality of H, we may assume that π embeds all but one edge, say e, of H into G. Denote the tail of e by w_0. Following the previous convention, the head of e is w_1, and the next two successive vertices after w_1 are w_2 and w_3. To simplify notation, we write $\pi(w_i) = v_i$, $i = 0,1,2,3$.

Since e is the only unembedded edge of H, x_0, x_1, x_2, x_3 and v_1, v_2, v_3 are paths in G, and v_0 and v_1 are not adjacent in G.

Throughout the proof, we consider e, w_0, π, and v_0 to be fixed. We shall choose y_0 so that the paths y_0, y_1, y_2, y_3 and w_0, w_1, w_2, w_3 have no edge in common. Hence, $w_3 \neq y_1, y_2$, or y_3, and $w_0 \neq y_1$ or y_2 (the case $w_0 = y_0$ is excluded by $w_3 \neq y_3$). For any other choice of $y_0 \in V(H)$, the two paths have no common edge. Thus, we have $p - 5$ choices for y_0. Each choice determines x_0, x_1, x_2, and x_3 since π is fixed.

For any choice of y_0, if $(v_1\ x_1)\pi$ or $(v_1\,x_1)(v_2\,x_2)\pi$ is an embedding, then H is a subgraph of G, and we are done. Otherwise, if neither is an embedding, then $E(G^c)$ includes some of the following six edges:

(8.2) $\{v_0,x_1\}$, $\{v_1,x_0\}$, $\{v_1,x_2\}$, $\{v_2,x_1\}$, $\{v_2,x_3\}$, $\{v_3,x_2\}$.

We shall estimate the number of values of y_0 for which $(v_2\ x_2)\pi$ embeds $H-e$ into G. Observe that for a given y_0 (which determines x_0,x_1,x_2,x_3),

(8.3) If exactly one of the six edges (8.2) is in $E(G^c)$, then it must be $\{v_0,x_1\}$ or $\{v_1,x_0\}$ (otherwise, $(v_1\ x_1)\pi$ or $(v_1\ x_1)(v_2\ x_2)\pi$ embeds H into G), and so $(v_2\ x_2)\pi$ embeds $H-e$ into G.

Let n_1 be the number of values of y_0 such that (8.3) holds. This leaves $p-5-n_1$ choices of y_0 such that at least 2 of the 6 edges (8.2) lie in $E(G^c)$. We count occurrences of edges in $E(G^c)$ among the 6 edges of (8.2) in two ways, as y_0 runs over $p-5$ vertices in H. It is clear that their number is at least

$$1(n_1) + 2(p-5-n_1) = 2p-10-n_1.$$

Also, each of at most $\Delta(G^c)$ edges incident with v_i is counted once among the 6 edges (8.2), if $i=0$ or 3, and each is counted twice if $i=1$ or 2. Hence, the number of edge-occurrences is at most $6\,\Delta(G^c)$.

Counting two ways, we get

$$2p - 10 - n_1 \leq \text{number of edge occurrences}$$
$$\leq 6\,\Delta(G^c).$$

Hence,

$$n_1 \geq 2p - 10 - 6\,\Delta(G^c),$$

and so by (8.1),

$$n_1 \geq \Delta(G^c) + 1.$$

Thus, there are at least $\Delta(G^c) + 1$ values of y_0, and hence $\Delta(G^c) + 1$ values of x_2, such that $(v_2\ x_2)\pi$ embeds $H - e$ into G.

The number of vertices x such that both vertices of $M(x)$ are adjacent in G to v_1 (i.e., the number of predecessors of v_1) is at least

$$p - 2\,\Delta(G^c) \geq \frac{3p + 22}{7}.$$

At most $(2p - 11)/7$ of these are not adjacent to v_0, and so the number of predecessors of v_1 that are adjacent to v_0 in G is at least $\frac{p + 11}{7}$. Let x be any one of these. Among the $\Delta(G^c) + 1$ values of x_2 such that $(v_2\ x_2)\pi$ embeds $H - e$ into G, choose x_2 to be adjacent to x. Then $(v_1\ x)(v_2\ x_2)\pi$ embeds H into G. But this contradicts the assumption that H is not a subgraph of G. The proof is complete.

9. Subgraphs with triangular components

In section 7 we gave two classes of graphs, denoted $C_1(d)$ and $C_2(d)$, such that if

$$(d+1)(d'+1) = p+2,$$

if $H \in C_1(d') \cup C_2(d')$, and if $G^c \in C_1(d) \cup C_2(d)$, then either H is not a subgraph of G or both $H \in C_2(d')$ and $G^c \in C_2(d)$. We conjectured that if

$$(\Delta(G^c)+1)(\Delta(H)+1) \leq p+1,$$

then H is a subgraph of G. Thus, these two classes C_1 and C_2 make the conjecture best possible.

To simplify notation in this section, we shall say that G is of type 1 or type 2 if $p = 3b+1$, $b > 0$, and either $G^c \in C_1(b)$ or $G^c \in C_2(b)$, respectively. Thus, when G is of type 1, there is a set S of $b-1$ vertices such that $G-S$ is isomorphic to $K_{b+1,b+1}$. Also, when G is of type 2, there is a stable set S of $b+1$ vertices such that $G-S$ has 2 components, both isomorphic to K_b, and b is odd.

Suppose $H \in C_1(2)$. If G is of type 1 or type 2, then clearly H is not a subgraph of G. We shall show that if $H \in C_1(2)$, then graphs G of types 1 and 2 are the <u>only</u> graphs with $\delta(G) \geq \frac{p-1}{3}$ such that H is not a subgraph of G.

<u>Theorem 9.1</u> Let G and H be graphs on p vertices, and suppose that every component of H is isomorphic to either K_1, K_2, or K_3. Let $b = b(H)$ denote the number of triangular components of H, and suppose $b \geq 0$. If

$$\delta(G) \geq [\frac{p+b}{2}],$$

and if H is not a subgraph of G, then either

(9.1) There is a set S of $b-1$ vertices of G such that $G-S$ is a complete bipartite graph; or

(9.2) There is a set S of $b+1$ vertices, b odd, such that $G-S$ has two components, both isomorphic to K_b, and H has $\frac{p-1}{3}$ triangles.

<u>Theorem 9.2</u> Let G and H be graphs on p vertices and suppose that every component of H is a triangle K_3, except for one vertex K_1 if $p = 3b+1$, or one edge K_2 if $p = 3b+3$. If

$$\delta(G) \geq \frac{2}{3}(p-1),$$

then H is not a subgraph of G if and only if both

$$\delta(G) = \frac{2}{3}(p-1) = 2b$$

and G is of type 1 or type 2.

If H is the graph of Figure 1, then Figures 2 and 3, respectively, are the complements of corresponding graphs of types 1 and 2 such that H is not a subgraph.

Lemma 9.3 Let G be a graph with $p = 3b + 1$ vertices, for some integer b, and with $\delta(G) \geq 2b$. If for some set $S \subseteq V(G)$, with $|S| = b - 1$, $G - S$ is bipartite, with bipartition $V_1 \cup V_2$, then the following conclusions hold:

Every vertex of S is adjacent to every vertex of $G - S$;

$|V_1| = |V_2|$;

$G - S$ is a complete bipartite graph.

Thus, G is of type 1.

Proof: Without loss of generality, assume that $|V_1| \geq |V_2|$. We have

$$|V_1| \geq \tfrac{1}{2}(p - |S|) = \tfrac{1}{2}(3b + 1 - (b - 1)) = b + 1.$$

Let $v_1 \in V_1$. Since $V_1 \cup V_2$ is a bipartition of $G - S$, v_1 is adjacent in G^c to every vertex of $V_1 - v_1$. But

$$\Delta(G^c) = p - \delta(G) - 1 \leq b,$$

and hence we must have $|V_1| = b + 1$ and $\delta(G) = 2b$. Also, each $v_1 \in V_1$ must be adjacent to every vertex of $G - V_1$ (i.e., to every vertex of V_2 and every vertex of S). The conclusions of the lemma follow directly.

Remarks: If G is of type 2, then $p \equiv 4 \pmod 6$, and G is regular of degree $2b = \frac{2}{3}(p - 1)$. Note that the only graph that is both of type 1 and type 2 is the quadrilateral.

<u>Lemma 9.4</u> Let G be a graph with $p = 3b+1$ vertices, for some integer b, and with $\delta(G) \geq 2b$. If for some set $S \subseteq V(G)$, with $|S| = b+1$, $G - S$ has two components, then the following conditions hold:

Every vertex of S is adjacent to every vertex of $G - S$;

$G - S$ has two components, both isomorphic to K_b.

If, furthermore, b pairwise disjoint triangles do not embed in G, then

$p \equiv 4 \pmod 6$;

S is a stable set;

G is of type 1 only if G is a quadrilateral.

Thus, G is of type 2.

<u>Proof</u>: Let G and S satisfy the hypotheses. Since $p = 3b+1$ and $\delta(G) \geq 2b$, any vertex is adjacent in G^c to at most b vertices of G. Thus, since $|V(G-S)| = 2b$ and since $G - S$ has two components, any vertex in the smaller component is adjacent in G^c to at least $\frac{1}{2}|V(G-S)| = b$ vertices in the larger component of $G - S$. But these statements force equality: both components have just b vertices. Also, the first two conclusions of the lemma follow immediately.

If S is not a stable set or if $p \not\equiv 4 \pmod 6$, then either $G[S]$ has an edge, or, since $p = 3b+1$, $p \equiv 1 \pmod 6$. In either case, an embedding of b pairwise disjoint triangles is easily found. The rest is easy.

<u>Proof of Theorem 9.1 from Theorem 9.2</u>: Assume without loss of generality that the components of H consist of b triangles K_3, $[\frac{p-3b}{2}]$ edges K_2, and $p-3b-2[\frac{p-3b}{2}]$ vertices K_1. By adding $[\frac{p-3b}{2}]$ vertices to H, each adjacent to both ends of a K_2, we can construct a graph H' on $p+[\frac{p-3b}{2}]$ vertices, where the components of H' consist of $b+[\frac{p-3b}{2}]$ triangles K_3 and $p-3b-2[\frac{p-3b}{2}]$ (= 0 or 1) vertices K_1. By adding a stable set of $[\frac{p-3b}{2}]$ vertices to G, we construct a graph G' in which each added vertex is adjacent to every vertex of G. Thus,

$$|V(G')| = p + [\frac{p-3b}{2}] = [3\frac{p-b}{2}],$$

and

$$\begin{aligned}
\delta(G') &\geq \min(p,\ \delta(G) + [\tfrac{p-3b}{2}]) \\
&\geq \min(p,\ [\tfrac{p+b}{2}] + [\tfrac{p-3b}{2}]) \\
&= \min(p,\ 2[\tfrac{p-b}{2}]) \\
&= 2[\tfrac{p-b}{2}] \\
&= \tfrac{2}{3}(3[\tfrac{p-b}{2}]) \\
&\geq \tfrac{2}{3}(|V(G')| - 1).
\end{aligned}$$

Thus, by Theorem 9.2, either H' is a subgraph of G', or G' is a graph of type 1 or type 2. Suppose G' is a graph of type 2. If $G' \neq G$, then G' has a vertex of degree p and $|V(G')| < \frac{3}{2}p$. Hence, G' is not a graph of type 2 unless $G' = G$. Then by Lemma 9.4,

$$|V(G')| = 4 \pmod 6.$$

In this case $[\frac{p-3b}{2}] = 0$ vertices were added to G to get G', whence $p - 3b = 1$, and H has $b = \frac{1}{3}(p-1)$ triangles, and we have the second case of Theorem 9.1.

Suppose G' is a graph of type 1. Then H' has

$$b' = b + [\frac{p-3b}{2}] = [\frac{p-b}{2}]$$

triangles. Moreover, $|V(G')| = 3b' + 1$, and there is a set $S' \subseteq V(G')$, with $|S'| = b' - 1$, whose removal leaves a complete bipartite graph $G' - S' = K_{b'+1,b'+1}$. We have

$$\delta(G') \geq 2[\frac{p-b}{2}] = 2b'.$$

We claim that $V(G') = V(G) \cup S'$. To prove this, suppose that $V(G) \cup S'$ does not contain a vertex $v \in V(G') - V(G)$. However, $V(G') - V(G)$ has only $[\frac{p-3b}{2}]$ vertices, and so some vertex w of G lies on the same side of the bipartition as v. But v is adjacent to all vertices of G, and in particular to w, and we have a contradiction, which proves the claim.

Let $S = V(G) \cap S'$. Then, by the claim,

$$\begin{aligned} |S| &= |S'| - (|V(G')| - |V(G)|) \\ &= (b + [\frac{p-3b}{2}] - 1) - [\frac{p-3b}{2}] \\ &= b - 1, \end{aligned}$$

and $G - S$ is bipartite. This is a conclusion of 9.1.

The remaining possibility is that H' is a subgraph of G'. There is an embedding of H' into G' which extends an embedding of H into G. This proves Theorem 9.1.

Lemma 9.5 Let G be a graph, and $X_1 \cup X_2$ be a partition of V(G) of the type described in Theorem 4.5, for which

(9.3) $\delta(G_1) + \delta(G_2) = \delta(G)$.

Suppose that sets $Y_3 \subseteq X_1$, $V_3 \subseteq X_2$ exist such that

(9.4) $G_2 - V_3$ is a complete bipartite graph with nontrivial bipartition $V_1 \cup V_2$;

(9.5) $G_1 - Y_3$ is a complete bipartite graph with nontrivial bipartition $Y_1 \cup Y_2$;

(9.6) If $v \in V_1 \cup V_2$ then $\deg_{G_2}(v) = \delta(G_2)$;

(9.7) If $y \in Y_1 \cup Y_2$ then $\deg_{G_1}(y) = \delta(G_1)$.

Then any vertex of $Y_1 \cup Y_2$ is adjacent to every vertex in V_j, for some $j \in \{1,2\}$. Suppose further that

(9.8) No vertex of $Y_1 \cup Y_2$ is adjacent to vertices in both V_1 and V_2.

Then $G - (Y_3 \cup V_3)$ is a complete bipartite graph.

Proof: By (9.3), (9.6), and (9.7), the latter part of Theorem 4.5 may be applied to the vertices of $V_1 \cup V_2 \cup Y_1 \cup Y_2$.

Suppose that the first conclusion of the lemma is false for some $y \in Y_1 \cup Y_2$. Thus, y is not adjacent in G to a vertex v_1 of V_1 and a vertex v_2 of V_2. By Theorem 4.5, v_1 and v_2 are interchangeable with y, and are thus not adjacent in G. But, by (9.4), v_1 is adjacent to v_2. This contradiction proves the first part of the lemma.

By (9.8), any vertex $y \in Y_1 \cup Y_2$ is adjacent in G^c to every vertex of V_j for $j \in 1,2$. By the first part of the lemma, which was just proved, y is adjacent to every vertex of V_{3-j}.

Thus, the vertices of $Y_1 \cup Y_2$ fall into two classes: those, the set of which we denote Y_4, which are adjacent to vertices of V_1 but not V_2; and those the set of which we denote Y_5, which are adjacent to vertices of V_2 but not V_1.

We claim that $\{Y_4, Y_5\} = \{Y_1, Y_2\}$. To see this, suppose that $Y_4 \cap Y_1$ and $Y_4 \cap Y_2$ are both nonempty. Then any vertex $v_2 \in V_2$ is not adjacent to a vertex $y_1 \in Y_4 \cap Y_1$, nor to a vertex $y_2 \in Y_4 \cap Y_2$. By Theorem 4.5, y_1 and y_2 are interchangeable with v_2 and are thus not adjacent. However, (9.5) implies that y_1 and y_2 are adjacent.

This contradiction shows that either $Y_4 \cap Y_1$ or $Y_4 \cap Y_2$ is empty. Similarly, either $Y_5 \cap Y_1$ or $Y_5 \cap Y_2$ is empty. Since $V_1 \cup V_2$ and $Y_1 \cup Y_2$ are nontrivial, and

$$Y_4 \cup Y_5 = Y_1 \cup Y_2,$$

the claim must follow.

In either case of this claim, there is a $j \in \{1,2\}$ such that $(V_1 \cup Y_j) \cup (V_2 \cup Y_{3-j})$ is a bipartition of $G - (Y_3 \cup V_3)$, and this bipartite graph is complete. This proves Lemma 9.5.

We define for $X_j' \subseteq V(G)$

$$G_j' = G[X_j'] \qquad j = 1,2,$$

and

$$p_j' = |X_j'| \qquad j = 1,2.$$

A vertex x of G, G_j or G_j' is <u>critical</u> in G, G_j, G_j', if

$$\deg_G(x) - 1 < \tfrac{1}{3}(p-1),$$

$$\deg_{G_j}(x) - 1 < \tfrac{1}{3}(p_j - 1),$$

or

$$\deg_{G_j'}(x) - 1 < \tfrac{1}{3}(p_j' - 1),$$

respectively.

Lemma 9.6 Suppose Theorem 9.2 is valid for all graphs with less than p vertices. Suppose

$$p \equiv 1 \pmod 3$$

and that $X_1 \cup X_2$ is a partition of $V(G)$ which satisfies the conditions of Theorem 4.5 with $c = \frac{2}{3}$. For $\{z, z'\} \subseteq V(G)$, write

$$X_j' = X_j - \{z, z'\} \qquad \text{for } j = 1, 2,$$

and assume that

$$p_j' \equiv 1 \pmod 3 \qquad \text{for } j = 1, 2,$$

$$(9.9) \quad \delta(G') \geq \tfrac{2}{3}(p_j' - 1) \qquad \text{for } j = 1, 2,$$

and that $p_j \equiv 0 \pmod 3$ for $j \in \{1, 2\}$ implies that $z \in X_j$ and that there exist critical vertices $x_3, x_4 \in X_{3-j}$ such that $G[z, x_3, x_4]$ is a triangle. Then, if $\frac{1}{3}(p_j' - 1)$ pairwise disjoint triangles cannot be embedded in G_j', for $j = 1$ and $j = 2$, both G_1' and G_2' are of type 1.

Proof: Since Theorem 9.2 holds for graphs on fewer than p vertices, since (9.9) holds, and since $\frac{1}{3}(p_j' - 1)$ triangles cannot be embedded in G_j', $j = 1, 2$, it follows that G_1' is of type 1 or type 2, and G_2' is of type 1 or type 2. Thus,

$$\delta(G_j') = \tfrac{2}{3}(p_j' - 1) \qquad j = 1, 2,$$

whence

$$(9.10) \quad \delta(G_1') + \delta(G_2') = \tfrac{2}{3}(p_1' + p_2' - 2)$$
$$= \tfrac{2}{3}(p - 1) - 2.$$

Moreover, by Theorem 4.5,

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1) - \frac{2}{3}.$$

The left side is an integer and $p \equiv 1 \pmod 3$, whence

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1).$$

So that (9.10) also holds, it follows that if z or z', respectively, is in X_j, for $j \in \{1,2\}$, then z or z' is adjacent in G to every critical vertex of G_j'. In fact

$$(9.11) \quad \delta(G_1) + \delta(G_2) = \frac{2}{3}(p-1),$$

and vertices critical in G_j' are critical in G_j, for $j = 1,2$. Also, since critical vertices of G_1 and critical vertices of G_2 are interchangeable if they are adjacent in G^c, critical vertices of G_1' and critical vertices of G_2' are also interchangeable if they are adjacent in G^c. By (4.19) of Theorem 4.5, such vertices are also critical in G.

Let y_1, y_2 be any pair of adjacent critical vertices of G_1'. If G_1' is of type 2, then every vertex of G_1' is critical in G_1', whence, any adjacent pair suffices. If G_1' is of type 1, then $p_1' \geq 4$, and there is a set $Y_3 \subseteq X_1'$ with

$$|Y_3| = \frac{1}{3}(p_1' - 1) - 1,$$

such that $G_1' - Y_3$ is a complete bipartite graph with bipartition $Y_1 \cup Y_2$, where

$$|Y_1| = |Y_2| = \frac{1}{3}(p_j' - 1) + 1.$$

Since $Y_1 \cup Y_2$ is the set of critical vertices in G_1', if $y_1 \in Y_1$ and $y_2 \in Y_2$, then y_1 and y_2 are adjacent critical vertices of G_1'.

Suppose by way of contradiction that G_2' is of type 2 and not of type 1. Then every vertex v of G_2 is critical in G_2, and hence interchangeable with y_i $(i = 1,2)$ if y_i is adjacent in G^c to v.

Since y_i is critical in G_1' and in G,

$$\begin{aligned} |E(y_i, X_2')| &= \deg_G(y_i) - \deg_{G_1'}(y_i) - |E(y_i, \{z, z'\})| \\ &= \frac{2}{3}(p - 1) - \frac{2}{3}(p_1' - 1) - |E(y_i, \{z, z'\})| \\ &= \frac{2}{3}(p_2' + 2) - |E(y_i, \{z, z'\})| . \end{aligned}$$

Hence, the number of vertices of G_2' adjacent in G^c to y_i is at least

$$p_2' - |E(y_i, X_2')| \geq \frac{1}{3}(p_2' - 1) + |E(y_i, \{z, z'\})| - 1,$$

and these vertices are interchangeable with y_i and thus form a stable set.

We have two cases: when $\{z, z'\} \cap X_1$ is not empty, and when $\{z, z'\} \subseteq X_2$. In the first case, without loss of generality, suppose $z \in X_1$. In G_1, z is adjacent to every critical vertex of G_1', including $y_1, y_2 \in X_1'$. Hence, $|E(y_i, \{z, z'\})| \geq 1$. In the second case, by the hypotheses of the lemma, $p_2 \equiv 0 \pmod 3$ and z lies in a triangle $G[z, x_3, x_4]$, where x_3 and x_4 are adjacent critical vertices of G_1. Pick y_1, y_2 so that $\{y_1, y_2\} = \{x_3, x_4\}$, which is possible because $G_1 = G_1'$ here. Then $|E(y_i, \{z, z'\})| \geq 1$. Therefore, in either case there are at least $\frac{1}{3}(p_2' - 1)$ critical vertices of G_2' interchangeable with y_i $(i = 1, 2)$.

Since G_2' is of type 2, $p_2' \equiv 4 \pmod 6$, and since G_2' is not of both type 1 and type 2, it follows that $p_2' \geq 10$. Hence, at least $\frac{1}{3}(p_2' - 1) \geq 3$ critical vertices of G_2' are interchangeable with y_i $(i = 1, 2)$. By Theorem 4.5, this set of $\frac{1}{3}(p_2' - 1)$ vertices is a stable set. But since G_2' is of type 2, there is only one maximal stable set S_2 of more than 2 vertices, and S_2 has $\frac{1}{3}(p_2' - 1) + 1$ vertices. Therefore, y_1 is interchangeable with all but at most one vertex of S_2. Since $|S_2| \geq 3$, there is a critical vertex $v \in S_2$, critical in G_2, interchangeable with both vertices y_1, y_2 critical in G_1. By Theorem 4.5, y_1 and y_2 are not adjacent, contrary to the choice of y_1 and y_2. Hence, G_2' is of type 1, and the lemma is proved.

We leave to the reader the proofs of the next two lemmas.

Lemma 9.7 Let G_0 be a graph of type 1 on $3b_0+1$ vertices. Let S_0 be the set of b_0-1 vertices whose removal leaves $G-S_0 = K_{b_0+1,b_0+1}$. Any embedding of b_0-1 pairwise disjoint triangles into G_0 uses all but four vertices $v_1,v_2,v_3,v_4 \in V(G_0)-S_0$, and these four vertices induce a quadrilateral in G_0. Furthermore, v_1,v_2,v_3,v_4 may be chosen to be any four vertices of G_0-S_0 that induce a quadrilateral in G.

Lemma 9.8 Let G_0 be a graph of type 2 on $3b_0+1$ vertices. Let S_0 be the stable set of b_0+1 vertices such that G_0-S_0 consists of two components, each K_{b_0}. Any embedding of b_0-1 pairwise disjoint triangles into G_0 uses all but four vertices, two in S_0, and one in each K_{b_0}, and these four vertices induce a quadrilateral in G_0. Furthermore, for any four vertices of $V(G_0)$ with two in S_0 and one in each K_{b_0}, there is an embedding of b_0-1 pairwise disjoint triangles into the remaining $3b_0-3$ vertices of G_0.

To save work, we assume without proof the following result of Corradi and Hajnal [7]:

Theorem 9.9 Let G and H be graphs on p vertices such that every component of H is a triangle, except possibly for one component that is either K_1 or K_2. If

$$\delta(G) \geq \frac{2p-1}{3},$$

then H is a subgraph of G.

Proof of Theorem 9.2: By Theorem 9.9, it suffices to consider graphs G for which

$$\delta(G) = \frac{2}{3}(p-1),$$

Equality implies that

$$p \equiv 1 \pmod 3.$$

Thus, we can assume that H is a graph with b triangles and one isolated vertex, and that

$$p = 3b+1,$$

$$\delta(G) = \frac{2}{3}(p-1) = 2b.$$

By Theorem 4.5 there are disjoint nonempty sets X_1, X_2 such that $V(G) = X_1 \cup X_2$ and the induced subgraphs G_i, for $G_i = G[X_i]$, $i = 1,2$, satisfy

$$(9.12) \quad \delta(G_i) \geq \frac{2}{3}(p_i - 1),$$

where $p_i = |X_i|$.

Assume inductively that Theorem 9.2 is true for graphs smaller than G, and suppose that H is not a subgraph of G. Theorem 9.2 is true for $p \leq 4$, and so we have a basis for induction. We have two cases: either one of the sets X_i has cardinality a multiple of 3, or neither do. In one subcase (Subcase IIA), we show that if H is not a subgraph of G, then G is of type 2. In other subcases, we verify the hypotheses of Lemma 9.5, and hence there is a subset $S = Y_3 \cup V_3$ of V(G), with $|S| = b - 1$, such that $G - S$ is a bipartite graph. Thus, by Lemma 9.3, G is of type 1. We consider each case below.

Case I: Suppose that

$$p_1 \equiv 0 \pmod 3)$$

and

$$p_2 \equiv 1 \pmod 3).$$

Since $\delta(G_1)$ is an integer and $p_1 \equiv 0 \pmod 3$, (9.12) gives

$$\delta(G_1) \geq \frac{2}{3}(p_1 - 1) + \frac{2}{3} = \frac{2}{3}p_1,$$

and Theorem 9.9 implies that $p_1/3$ triangles can be embedded in G_1. Write

$$b_1 = \frac{1}{3}p_1$$

and

$$(9.13) \quad b_2 = \frac{1}{3}(p_2 - 1),$$

and note that

$$b_1 + b_2 = b,$$

and that the $b_1 \geq 1$ triangles embedded in G_1 use each vertex of G_1. Since b triangles are assumed to not embed in G, it follows that b_2 triangles do not embed in G_2. By the induction hypothesis, either G_2 is of type 1, and there is a set $V_3 \subseteq X_2$ with

(9.14) $|V_3| = b_2 - 1$

such that

$$G_2 - V_3 = K_{b_2+1, b_2+1},$$

or G_2 is of type 2 and there is a stable set

(9.15) $S_2 \subseteq X_2$

such that $G_2 - S_2$ has two components, each a clique on b_2 vertices.

If G_2 is of type 2, each vertex $v \in X_2$ has degree

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1),$$

and so (9.6) holds. If this alternative applies, write

(9.16) $V_2 = S_2$, $V_1 = G_2 - S_2$.

If G_2 is of type 1, let

(9.17) $V_1 \cup V_2$ denote the bipartition of $G_2 - V_3$.

Then

$$|V_1| = |V_2| = b_2 + 1,$$

and (9.12) and (9.13) give

$$\delta(G_2) \geq 2b_2,$$

which allows us to apply Lemma 9.3. Also, by Lemma 9.3, each vertex of V_j $(j=1,2)$ is adjacent to every vertex of V_{3-j} and to every vertex of V_3, and if $v \in V_1 \cup V_2$,

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1)$$

whence (9.6) holds.

It follows in either alternative (either G_2 of type 1 or type 2) that there must be at least

$$\begin{aligned} \deg_G(v) - \deg_{G_2}(v) &\geq \frac{2}{3}(p-1) - \frac{2}{3}(p_2 - 1) \\ &= \frac{2}{3}p_1 \end{aligned}$$

vertices in X_1 adjacent to a given vertex $v \in V_1 \cup V_2$.

Denote by $N(v_1, v_2)$ the vertices of X_1 that are adjacent to both $v_1 \in V_1$ and $v_2 \in V_2$. We have

(9.18) $|N(v_1, v_2)| \geq 2(\frac{2}{3}p_1) - p_1 = \frac{1}{3}p_1 = b_1$.

Since G_2 is of type 1 or type 2, b_2 disjoint triangles do not embed in G_2. By Lemmas 9.7 and 9.8, there is an embedding of $b_2 - 1$ pairwise disjoint triangles into G_2 such that the four remaining vertices induce a quadrilateral in G_2, with two of its vertices in V_1 and the other two in V_2. Let $\{v_1, v_2\}$ and $\{v_1', v_2'\}$ be disjoint edges of this quadrilateral, where $v_1, v_1' \in V_1$, and $v_2, v_2' \in V_2$.

In the two subcases below, we establish that G_2 is of type 1, and that the hypotheses of Lemma 9.5 apply to G_1 and G_2. We have already established (9.4) and (9.6), and it remains to establish (9.3), (9.5), and (9.7). After the subcases, we prove (9.8).

Subcase IA: Suppose $v_1, v_1' \in V_1$ are distinct and $v_2, v_2' \in V_2$ are distinct. Suppose that $N(v_1, v_2)$, $N(v_1', v_2')$ possess a transversal $\{y, y'\}$ in X_1; i.e., distinct $y, y' \in X_1$ such that

$$y \in N(v_1, v_2), \quad y' \in N(v_1', v_2')$$

Since

$$\delta(G_1) \geq \frac{2}{3} p_1,$$

we have

$$\begin{aligned} \delta(G_1 - \{y, y'\}) &\geq \frac{2}{3} p_1 - 2 \\ &= \frac{2}{3}(p_1 - |\{y, y'\}| - 1). \end{aligned}$$

Since b pairwise disjoint triangles do not embed in G, and since $(b_2 - 1) + 2$ triangles can be embedded in $G[X_2 \sim \{y, y'\}]$, we cannot embed

$$b - (b_2 + 1) = b_1 - 1$$

triangles in $G_1 - \{y, y'\}$. By the induction hypotheses, $G_1 - \{y, y'\}$ is a graph of type 1 or of type 2, and by Lemma 9.6 with $\{y, y'\} = \{z, z'\}$, and with $\{v_1, v_2\} = \{x_3, x_4\}$, both $G_1 - \{y, y'\}$ and G_2 are of type 1. Therefore, there is a set Y_3' of $b_1 - 2$ vertices such that

$G_1 - Y_3$ is bipartite, where

(9.19) $\quad Y_3 = Y_3' \cup \{y, y'\}$.

Let $Y_1 \cup Y_2$ be the bipartition of $G_1 - Y_3$. By definition,

$$|Y_1| = |Y_2| = b_1 = \frac{1}{3} p_1,$$

and by Lemma 9.3, each vertex y_j of Y_j $(j = 1,2)$ is adjacent to every vertex of $Y_{3-j} \cup Y_3'$ and has degree $\frac{2}{3} p_1 - 2$ in $G_1 - \{y, y'\}$. Thus, (9.5) holds. Since

$$\deg_{G_1}(y_j) \geq \delta(G_1) = \frac{2}{3} p_1,$$

each vertex of Y_j is also adjacent to y and y', and hence has degree $\delta(G_1)$ in G_1, whence we have (9.7). Therefore,

$$\begin{aligned} \delta(G_1) + \delta(G_2) &= \frac{2}{3} p_1 + \frac{2}{3}(p_2 - 1) \\ &= \frac{2}{3}(p - 1) \\ &= \delta(G), \end{aligned}$$

which is (9.3). We have thus proved (9.3) and (9.4) through (9.7) of Lemma 9.5. This completes Subcase IA.

Subcase IB: Suppose that there is no pair of disjoint edges $\{v_1, v_2\}$, $\{v_1', v_2'\}$ in $G_2[V_1 \cup V_2]$ such that $N(v_1, v_2)$, $N(v_1', v_2')$ possess a transversal.

Since $p_1 > 0$, (9.18) implies that $b_1 \geq 1$ and that $N(v_1, v_2)$ and $N(v_1', v_2')$ are nonempty. Since $N(v_1, v_2)$, $N(v_1', v_2')$ possess no transversal, we have $y \in X_1$ such that

$$N(v_1, v_2) = y = N(v_1', v_2').$$

Hence, x_1 is not adjacent to itself in G_1, nor to

$$\frac{1}{3}(p-1) = \frac{1}{3}(p_2-1) + 1$$

vertices in X_2, which, by Theorem 4.5, must be a stable set. Since G_2 is of type 2, there is only one stable set, namely S_2, by (9.15), of

$$b_2 + 1 = \frac{1}{3}(p_2 - 1) + 1$$

vertices, unless $b_2 + 1 = 2$. If $b_2 = 1$, then $p_2 = 4$, and G_2 is a quadrilateral, which is also of type 1. If $b_2 \geq 2$, each vertex of X_1 is interchangeable with any vertex of S_2, and since they are interchangeable, Theorem 4.5 implies that X_1 is stable. This contradicts the fact that G_1 is a triangle. Hence, G_2 is of type 1.

Finally, we must show that (9.8) of Lemma 9.5 applies in either subcase. Let y, $N(v_1, v_2)$, and $N(v_1', v_2')$ be as in the subcases above. Suppose (9.8) is false.

There exists a vertex $y'' \in Y_1 \cup Y_2$ that forms a triangle with vertices of $V_1 \cup V_2$. Then the first part of Lemma 9.4 implies that y" is adjacent to all vertices of V_j and to some of the vertices of V_{3-j}, for $j = 1$ or 2. Choose

$$v = v_{3-j} \text{ or } v_{3-j}'$$

so that y'' is adjacent to a vertex v_{3-j} of $V_{3-j} - v$. Without loss of generality, suppose

$$v = v_{3-j}.$$

Then, v_{3-j}, v_j and y form a triangle, and y'' forms a triangle with v_j' and v_{3-j}''. Thus, there are two disjoint triangles, which together with the $b_2 - 1$ triangles that can be embedded in $G_2 - \{v_j, v_j', v_{3-j}, v_{3-j}'\}$, and the $b_1 - 1$ triangles that can be embedded in $G_1 - \{y'', y\}$ constitute an embedding of

$$2 + b_2 - 1 + b_1 - 1 = b$$

triangles in G. We have contradicted the nonembeddability assumption, and hence (9.8) is true. Thus, all of the hypotheses of Lemma 9.5 hold. We conclude from Lemma 9.5 that $G - (Y_3 \cup V_3)$ is a bipartite graph. By (9.14), (9.19), (9.20), we have

$$|Y_3 \cup V_3| = b - 1,$$

and so by Lemma 9.3, G is of type 1. This completes Case I.

Case II: Suppose that

$$p_1 \equiv p_2 \equiv 2 \pmod 3).$$

Since $\delta(G_i)$ is an integer, (9.12) implies

$$(9.22) \quad \delta(G_i) \geq \frac{2}{3}p_i - \frac{1}{3}$$

for $i = 1,2$. Without loss of generality, assume

$$b_1 \leq b_2.$$

Write

$$b_1 = \frac{1}{3}p_1 - \frac{2}{3}, \quad b_2 = \frac{1}{3}p_2 - \frac{2}{3},$$

and note that b_1 and b_2 are integers such that

$$b_1 + b_2 + 1 = b.$$

If we form a graph $G_i + z$, adding to G_i $(i = 1,2)$ a new vertex z adjacent to every vertex of G_i, then by (9.22),

$$\delta(G_i + z) = \frac{2}{3}|X_i + z|,$$

and by Theorem 9.9, $b_i + 1$ pairwise disjoint triangles can be embedded in $G_i + z$. Therefore, b_i pairwise disjoint triangles and an edge disjoint from the b_i triangles, which we shall call the free edge, can be embedded in G_i. We shall attempt to use the vertices of the two free edges to form an extra triangle, disjoint from the b_1 triangles in G_1 and the b_2 triangles in G_2, thus constituting $b_1 + b_2 + 1 = b$ pairwise disjoint triangles in G. By assuming that b pairwise disjoint triangles do not embed in G, we shall determine the structure of G in the attempt to find such an embedding.

We show in the two subcases below that either G is of type 2, or there is a vertex $x_3 \in X_2$ such that the free edge in G_1 together with x_3 form a triangle in G. It may be necessary to alter the embedding of b_1 triangles and the free edge into G_1 in order to accomplish this.

Let x_1, x_2 be the ends of the free edge in G_1. Without loss of generality, choose the free edge from among all possible free edges so that

$$\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$$

is minimized. If x_1 and x_2 are adjacent in G to a vertex $x_3 \in X_2$, then x_1, x_2, x_3 is the desired triangle. Otherwise, x_1 and x_2 are adjacent to no common vertex in X_2. Then

$$(9.23) \quad \begin{aligned} \deg_{G_1}(x_1) + \deg_{G_1}(x_2) &\geq 2\delta(G) - p_2 \\ &\geq \frac{4}{3}(p-1) - p_2 \\ &= p_1 + \frac{1}{3}p - \frac{4}{3}. \end{aligned}$$

Also, without loss of generality, assume that

$$\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2).$$

These inequalities imply

$$(9.24) \quad \begin{aligned} 2\deg_{G_1}(x_1) &\geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \\ &\geq p_1 + \frac{1}{3}p - \frac{4}{3}. \end{aligned}$$

We define

$$\pi\colon V(H_1) \longrightarrow V(G_1)$$

to be an embedding of b_1 triangles K_3 and one edge-component K_2, constituting H_1, into G_1 such that the edge-component K_2 is mapped to the free edge x_1, x_2 that minimizes $\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$. We shall alter π if necessary, and then either we shall extend π to an embedding of H into G, where H consists of b triangular components and one isolated vertex, or we shall show (Subcase IIA) that G is of type 2 or (following the subcases) that G is of type 1.

Define

$$M(x) = \{x' \in X_1 : \pi^{-1}(x) \text{ and } \pi^{-1}(x') \text{ are adjacent in } H_1\}.$$

For $i = 1,2$, and $x \in V(G)$, define

$$N_i(x) = \{x' \in X_i : \ x \text{ and } x' \text{ are adjacent in } G\}.$$

We say that $x \in X_1$ is a <u>successor</u> of $x_1 \in X_1$ if each vertex of $M(x_1)$ is adjacent in G_1 to x. Denote the set of successors of x_1 by $S(x_1)$. We say that $x_1 \in X_1$ is a <u>predecessor</u> of $x \in X_1$ if x is a successor of x_1. Denote the set of predecessors of x by $P(x)$.

Recall from section 1 that if $x_1, x_4 \in X_1$ are equal, then $(x_1\ x_4)'$ is the identity permutation on X_1, but if x_1, x_4 are distinct, then $(x_1\ x_4)' = (x_1\ x_4)$.

<u>Subcase IIA</u>: Suppose that

$$\deg_{G_1}(x_2) \le \frac{1}{3}(p-1).$$

First, we eliminate the possibility of strict inequality. If the inequality above is strict, then

$$\begin{aligned}|E(x_2,X_2)| &= \deg_G(x_2) - \deg_{G_1}(x_2)\\ &> \frac{2}{3}(p-1) - \frac{1}{3}(p-1)\\ &= \frac{1}{3}(p-1).\end{aligned}$$

Since x_1 is not adjacent to at most $\frac{1}{3}(p-1)$ vertices of G other than itself, x_1 is adjacent to one of the more than $\frac{1}{3}(p-1)$ vertices x_3 of X_2 incident with an edge of $E(x_2,X_2)$. Hence, $G[x_1,x_2,x_3]$ is a triangle on the free edge in G_1 and a vertex of G_2.

Henceforth in this subcase, we shall suppose

$$\deg_{G_1}(x_2) = \frac{1}{3}(p-1).$$

By (9.23),

$$\deg_{G_1}(x_1) + \frac{1}{3}(p-1) \ge p_1 + \frac{1}{3}p - \frac{4}{3}.$$

Hence,

$$\deg_{G_1}(x_1) \ge p_1 - 1,$$

and so x_1 must be adjacent to each vertex of G_1. Therefore, $P(x_1) = G_1 - x_2$. Since $S(x_1) = N_1(x_2)$, we conclude that for any $x_4 \in N_1(x_2)$, $(x_1\ x_4)'\pi$ is an embedding of the b_1 triangles and free edge into G_1. Note that

the embedding $(x_1\ x_4)'\pi$ makes $\{x_4, x_2\}$ the free edge.
By the minimality of $\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$,

$$\deg_{G_1}(x_4) + \deg_{G_1}(x_2) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2),$$

whence,

$$\deg_{G_1}(x_4) = p_1 - 1.$$

Since x_4 may be any of the $\frac{1}{3}(p-1)$ vertices of $N_1(x_2)$, we know that the vertices of $X_1 - N_1(x_2)$ must be adjacent to each vertex of $N_1(x_2)$, a set of $\frac{1}{3}(p-1)$ vertices adjacent to all of G_1. Hence,

$$\delta(G_1) \geq \tfrac{1}{3}(p-1) = \deg_{G_1}(x_2).$$

Define the sets

$$T_1 = N_1(x_2),$$
$$T_2 = N_2(x_2),$$
$$S_1 = X_1 - T_1,$$
$$S_2 = X_2 - T_2.$$

We have already shown that $G[T_1]$ is a complete graph, and each vertex of S_1 is adjacent to every vertex of T_1. If there is an $x_4 \in T_1 = S(x_1)$ and a vertex $x_3 \in X_2$ such that $G[x_2, x_3, x_4]$ is a triangle in G, then we have accomplished the goal of this subcase, since $(x_1\ x_4)'\pi$ is an embedding of b_1 triangles and a disjoint edge mapped to $\{x_2, x_4\}$, which is the edge forming the triangle with x_3. Otherwise, no $x_4 \in T_1$ forms a triangle with x_2 and any vertex in X_2. Hence, no $x_4 \in T_1$ is adjacent to vertices of T_2. Now,

$$|T_2| = \deg_G(x_2) - \deg_{G_1}(x_2) \geq \frac{1}{3}(p-1),$$

and hence, any $x_4 \in T_1$, having degree at least $\frac{2}{3}(p-1)$ in G, must be adjacent to every vertex of $S_1 \cup T_1 \cup S_2 - x_4$. A similar argument shows that any vertex of T_2, not being adjacent to any vertex of T_1, a set of $\frac{1}{3}(p-1)$ vertices, is adjacent to any vertex of $S_1 \cup T_2 \cup S_2$ except itself. Note that this implies that $G[T_2]$ is, like $G[T_1]$, a complete graph on $\frac{1}{3}(p-1)$ vertices. Also, note that any vertex of $S_1 \cup S_2$ is adjacent to every vertex of $T_1 \cup T_2$ in G.

Hence, $S_1 \cup S_2$ is a set of

$$\begin{aligned} |V(G) - (T_1 \cup T_2)| &= p - \frac{2}{3}(p-1) \\ &= \frac{1}{3}(p-1) + 1 \\ &= b + 1 \end{aligned}$$

vertices whose removal from G leaves two components $G[T_i]$, $i = 1,2$, each a complete graph on $\frac{1}{3}(p-1) = b$ vertices.

By Lemma 9.4, either b pairwise disjoint triangles embed in G, or G is of type 2. The first possibility is contrary to hypothesis. The other possibility is a desired conclusion of Theorem 9.1. Hence, we can assume that there is a free edge in G_1, which together with some $x_3 \in X_2$, forms a triangle in G.

<u>Subcase IIB:</u> Suppose that

$$\text{(9.25)} \quad \deg_{G_1}(x_2) > \tfrac{1}{3}(p-1).$$

Let x_3 be a vertex of X_2 that is adjacent in G to x_2. Since $p_1 \le p_2$,

$$\begin{aligned}\deg_G(x_2) &\ge \tfrac{2}{3}(p-1)\\ &\ge \tfrac{2}{3}(2p_1-1)\\ &= p_1 + \tfrac{1}{3}p_1 - \tfrac{2}{3}\\ &> p_1 - 1,\end{aligned}$$

and so x_3 exists. The successors $S(x_1)$ of x_1 in G_1 are the vertices of G_1 adjacent to x_2. We see that $S_1(x_1) = N_1(x_2)$. We have

$$\begin{aligned}|S(x_1) \cap N_1(x_3)| &\ge \deg_{G_1}(x_2) + \deg_G(x_3)\\ &\quad - (p_2-1) - |S(x_1) \cup N_1(x_3)|\\ &\ge \deg_{G_1}(x_2) + \tfrac{2}{3}(p-1) - (p_2-1) - p_1\\ &= \deg_{G_1}(x_2) - \tfrac{1}{3}(p-1)\\ &> 0,\end{aligned}$$

by (9.25). Hence, there is a vertex $x_4 \in X_1$ that forms a triangle with x_2 and x_3 and is a successor of x_1.

If $x_1 \in S(x_4)$, then the embedding $(x_1\ x_4)\pi$ maps the free edge in G_1 to $\{x_2, x_4\}$, which forms with $x_3 \in X_2$ a triangle in G as desired. Otherwise,

$$\text{(9.26)} \quad x_1 \notin S(x_4).$$

We shall find a vertex $x_5 \in X_1$ with $x_5 \in S(x_4) \cap P(x_1)$, whence $(x_1\ x_4\ x_5)\pi$ is the desired embedding of b_1 triangles and one edge into G_1.

In the image of the triangle embedded into G_1 having vertex x_4 are two other vertices, which we call x_6, x_7. The successors of x_4 are those vertices in G_1 adjacent to both x_6 and x_7. Hence, $x_1, x_6, x_7 \notin S(x_4)$, and

(9.27) $\quad |S(x_4)| \geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1.$

The predecessors $P(x_1)$ of x_1 in G_1 are those vertices $v \in X_1$ such that x_1 is adjacent to all vertices of $M(v)$. Now, x_1 is adjacent in G_1^c to $p_1 - \deg_{G_1}(x_1) - 1$ vertices $v' \in X_1$. Any such v' lies in exactly two sets $M(v)$, $v \in X_1$. Thus, $x_1 \notin S(v)$ for at most

$$2p_1 - 2\deg_{G_1}(x_1) - 2$$

vertices v of $X_1 - M(x_1) = G - x_2$. Since the remaining vertices of $G_1 - x_2$ are in $P(x_1)$, we have $x_2 \notin P(x_1)$, and

(9.28) $$|P(x_1)| \geq |X_1 - x_2| - (2p_1 - 2\deg_{G_1}(x_1) - 2)$$
$$= 2\deg_{G_1}(x_1) - p_1 + 1$$
$$\geq \tfrac{1}{3}(p-1),$$

by (9.24).

Suppose first that x_4 is not adjacent to x_1. Then

$$x_1, x_6, x_7 \notin P(x_1),$$

and we combine (9.27), (9.28), (9.22), and $2p_1 \leq p$ to get

$$x_1, x_2, x_6, x_7 \notin S(x_4) \cap P(x_1);$$
$$x_6, x_7 \notin S(x_4) \cup P(x_1),$$

and

$$(9.29)\quad |S(x_4) \cap P(x_1)| \geq |S(x_4)| + |P(x_1)| - |X_1 - \{x_6, x_7\}|$$
$$\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \frac{p}{3} - \frac{1}{3} - p_1 + 2$$
$$\geq 2\delta(G_1) - 2p_1 + \frac{p}{3} + \frac{5}{3}$$
$$\geq 2(\tfrac{2}{3}p_1 - \tfrac{1}{3}) - 2p_1 + \frac{p}{3} + \frac{5}{3}$$
$$= \frac{p}{3} - \frac{2}{3}p_1 + 1$$
$$\geq 1.$$

Suppose, otherwise, that x_4 is adjacent to x_1. Then $G[x_1, x_2, x_4]$ is a triangle, and $\{x_6, x_7\}$ is a free edge. Thus, by choice of $\{x_1, x_2\}$ and (9.23),

$$(9.30)\quad \deg_{G_1}(x_6) + \deg_{G_1}(x_7) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2)$$
$$\geq p_1 + \frac{p}{3} - \frac{4}{3}.$$

We combine (9.27), (9.28), (9.30) and

$$p_1 + p_2 = p$$

to obtain

$$(9.31)\quad |S(x_4) \cap P(x_1)| \geq |S(x_4)| + |P(x_1)| - p_1$$
$$\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 + \frac{p}{3} - \frac{1}{3} - p_1$$
$$\geq p_1 + \frac{p}{3} - \frac{4}{3} - 2p_1 + \frac{p}{3} - \frac{1}{3}$$
$$= \frac{2p}{3} - \frac{2}{3}p_1 - \frac{1}{3}p_1 - \frac{5}{3}$$
$$\geq \frac{2}{3}p_2 - \frac{1}{3}p_1 - \frac{5}{3}$$
$$= \frac{1}{3}(p_2 - p_1) + (\tfrac{1}{3}p_2 - \tfrac{5}{3}).$$

Note that both of the terms in the last line of (9.31) are nonnegative if $p_2 \geq 5$, and if $p_2 > 5$, then the last line is positive. If $p_2 \leq 5$, then $p_1 \leq p_2$ and $p_i \equiv 2 \pmod 3$ imply one of the following three cases:

$$p_2 = p_1 = 5;$$

$$p_2 = 5, \quad p_1 = 2;$$

or

$$p_2 = p_1 = 2.$$

If $p_2 = p_1 = 5$, then (9.23) gives

$$\deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 7,$$

whence $\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2)$ implies that x_1 is adjacent to every vertex of G_1 except itself, whence $x_4 \in P(x_1)$, in violation of (9.26). If $p_2 = 5$, $p_1 = 2$, then the last line of (9.31) is 1, which is as desired. If $p_1 = p_2 = 2$, then $p = 4$ and $\delta(G) \geq \frac{2}{3}(p-1)$ imply G is K_4, $K_4 - e$ (e an edge), or a quadrilateral, all of which satisfy the theorem. Hence, under our hypotheses, the last line of (9.31) and the last line of (9.29) may be assumed to be positive.

Therefore, whether or not x_4 and x_1 are adjacent, there is a vertex $x_5 \neq x_1$ or x_2, such that

$$x_5 \in S(x_4) \cap P(x_1),$$

and so we have a closed alternating chain in G_1 represented by the permutation

$$\alpha = (x_1\ x_4\ x_5).$$

Hence, $\alpha\pi$ is an embedding of the b_1 triangles and one edge into G_1. The free edge is determined by $\alpha\pi$ to be $\{x_2, x_4\}$, since x_1 is permuted to x_4 and since $x_2 \neq x_5$ guarantees that x_2 is fixed. Thus, the free edge is part of a triangle $G[x_2, x_3, x_4]$, as desired. <u>This concludes Subcase IIB</u>.

To complete Case II and the proof of the theorem, we verify that all the hypotheses, and hence the final conclusion, of Lemma 9.5 apply to G_1 and G_2, and then we show that G is of type 1.

Since we have assumed that

$$b = b_1 + 1 + b_2$$

triangles do not embed in G, and since $b_1 + 1$ triangles embed in $G_1 + x_3 = G[X_1 + x_3]$, we know that we cannot embed b_2 triangles in $G_2 - x_3$. Now,

$$|V(G_2) - x_3| = 3b_2 + 1,$$

and by (9.22),

$$\begin{aligned} \delta(G_2 - x_3) &\geq \delta(G_2) - 1 \\ &\geq \frac{2}{3}p_2 - \frac{1}{3} - 1 \\ &= \frac{2}{3}(|X_2 - x_3| - 1) \\ &= 2b_2. \end{aligned}$$

Since

$$\delta(G) = 2b = 2b_1 + 2 + 2b_2,$$

and since

$$\deg_{G_2}(v_j) = 2b_2 + 1 \qquad (j = 1,2),$$

each v_j $(j = 1,2)$ is adjacent to at least $2b_1 + 1$ vertices of G_1. Hence, there are at least

$$\begin{aligned} |E(v_1,X_1)| + |E(v_2,X_1)| - p_1 &\geq 2(2b_1 + 1) - (3b_1 + 2) \\ &= b_1 \\ &\geq 1 \end{aligned}$$

choices $y_3 \in X_1$ such that $G[v_1,v_2,y_3]$ is a triangle. Therefore, as we already remarked, we may apply Lemma 9.6 and conclude that both $G_1 - y_3$ and $G_2 - x_3$ are of type 1.

Next, we establish the hypotheses of Lemma 9.5.

Since $G_1 - y_3$ and $G_2 - v_3$ are of type 1, where

$$v_3 = x_3,$$

and since they have $3b_1 + 1$, $3b_2 + 1$ vertices, respectively, there are sets $Y_3' \subseteq X_1 - y_3$ and $V_3' \subseteq X_2 - v_3$ with

$$|Y_3'| = b_1 - 1,$$

$$|V_3'| = b_2 - 1,$$

such that $G_1 - y_3 - Y_3'$ and $G_2 - v_3 - V_3'$ are complete bipartite graphs $Y_1 \cup Y_2$ and $V_1 \cup V_2$, respectively. Define

$$Y_3 = Y_3' + y_3,$$

$$V_3 = V_3' + v_3.$$

Therefore, by the induction hypothesis, $G_2 - x_3$ is of type 1 or type 2. Hence,

$$\delta(G_2 - x_3) = \frac{2}{3}(p_2 - 2) = \frac{2}{3}p_2 - \frac{4}{3},$$

whence, by (9.22), x_3 is adjacent to every vertex of $G_2 - x_3$ having degree $\frac{2}{3}p_2 - \frac{4}{3} = \delta(G_2 - x_3)$ in $G_2 - x_3$.

By Lemmas 9.7 and 9.8, we know that $b_2 - 1$ triangles embed in $G_2 - x_3$, and that such an embedding uses all but 4 vertices of $G_2 - x_3$. Moreover, these 4 vertices all have degree $\delta(G_2 - x_3)$ in $G_2 - x_3$, and they induce a quadrilateral. Now, x_3 is adjacent to all four of these vertices, and hence forms a triangle with 2 of them. Let v_1 and v_2 denote the other 2 vertices on this quadrilateral. Note that v_1 and v_2 are adjacent. We shall show that there are b_1 choices of a vertex $y_3 \in X_1$ such that $G[v_1, v_2, y_3]$ is a triangle. If b_1 disjoint triangles can be embedded in $G_1 - y_3$, then, counting the triangle containing x_3, the triangle $G[v_1, v_2, y_3]$, and the $b_2 - 1$ triangles of $G_2 - x_3$, we have b pairwise disjoint triangles in G, contrary to assumption. Hence, b_1 triangles do not embed in $G_1 - y_3$. For this to happen,

$$b_1 \geq 1.$$

Thus, by (9.22), we may apply Lemma 9.6, with

$$\{z, z'\} = \{x_3, y_3\},$$

and conclude that both $G_1 - y_3$ and $G_2 - x_3$ are of type 1.

Thus, (9.4) and (9.5) of Lemma 9.5 hold, and also

$$(9.32) \quad |Y_3 \cup V_3| = b_1 + b_2 = b - 1.$$

Since $G_1 - y_3$ is of type 1, if $y \in Y_1 \cup Y_2$, then

$$\deg_{G_1 - y_3}(y) = 2b_1 = \frac{2}{3}(p_1 - 2).$$

Now, $\delta(G_1) \geq \frac{2}{3}p_1 - \frac{1}{3}$, and hence y is adjacent to $y_3 \in X_1$. Therefore, for any $y \in Y_1 \cup Y_2$,

$$\deg_{G_1}(y) = \frac{2}{3}(p_1 - 2) + 1 = \delta(G_1),$$

and (9.7) of Lemma 9.5 is established. Similarly, since $G_2 - v_3$ is of type 1, (9.6) may be established, and also for any $v \in V_1 \cup V_2$,

$$\deg_{G_2}(v) = \delta(G_2).$$

By (9.22),

$$\begin{aligned} \delta(G_1) + \delta(G_2) &\geq \frac{2}{3}p_1 - \frac{1}{3} + \frac{2}{3}p_2 - \frac{1}{3} \\ &= \frac{2}{3}(p - 1) \\ &= \delta(G), \end{aligned}$$

and (9.3) is established. Thus, having proved (9.3) through (9.7) of Lemma 9.5, we conclude from Lemma 9.5 that any vertex $y \in Y_1 \cup Y_2$ is adjacent to every vertex in V_j for some $j \in \{1, 2\}$.

Suppose by way of contradiction that some $y \in Y_1 \cup Y_2$ is adjacent in G to vertices $v_1 \in V_1$ and $v_2 \in V_2$ (i.e., suppose that (9.8) is false). Thus, $G[y, v_1, v_2]$ is a triangle. By Lemma 9.7, for any vertices $v_5 \in V_1 - v_1$ and $v_4 \in V_2 - v_2$, there is an embedding of $b_2 - 1$ triangles into $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$, since $G_2 - v_3$ is of type 1. Note that $G[v_3, v_4, v_5]$ is also a triangle. We conclude from Lemma 9.7 that for any vertices $y_1 \in Y_1$, $y_2 \in Y_2$, there is an embedding of $b_1 - 1$ pairwise disjoint triangles into $G_1 - \{y, y_1, y_2, y_3\}$, since $G_1 - y_3$ is of type 1. Including the $b_2 - 1$ triangles of $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$ and the 3 triangles $G[y_1, y_2, y_3]$, $G[y, v_1, v_2]$, and $G[v_3, v_4, v_5]$, we have

$$(b_1 - 1) + (b_2 - 1) + 3 = b$$

pairwise disjoint triangles embedded in G, contrary to assumption. Hence, (9.8) holds, and by Lemma 9.5, $G - (Y_3 \cup V_3)$ is a complete bipartite graph. By (9.32) and Lemma 9.3, G is of type 1. This completes the proof of Theorem 9.2.

10. Subgraphs of graphs, III

We give one of our main results in this section.

Theorem 10.1 Let G and H be graphs on p vertices. If $\Delta(H) \leq 2$ and if

(10.1) $\quad \Delta(G^c) \leq \frac{p}{3} - \max(k, \frac{3}{2} p^{1/3}),$

where $k = 9$, then H is a subgraph of G.

Proof: The entire chapter is devoted to the proof of this Theorem. First, we introduce notation.

Recall that for a bijection

$$\pi: \ V(H) \longrightarrow V(G),$$

if $v \in V(G)$, then

$$M(v) = \{x: \pi^{-1}(x) \text{ and } \pi^{-1}(v) \text{ are adjacent in } H\}.$$

We shall use the notation

$$M(v_1, v_2, \ldots, v_n) = \bigcup_{i=1}^{n} M(v_i),$$

and

$$M(A) = \bigcup_{v \in A} M(v),$$

where the latter union is over the vertices $v \in A$, where $A \subseteq V(G)$.

Successors and predecessors are defined as in section 7, except that we do not permit vertices of $M(v)$ to be successors or predecessors of v. Thus, $x \in V(G)$ is a successor of $v \in V(G)$ if x is adjacent in G to each vertex of $M(v)$. The set of successors of v is denoted $S(v)$. Also, v is a predecessor of x whenever x is a successor of v. The set of predecessors of x is denoted $P(x)$.

Suppose that H is an edge-minimal graph for which the theorem is false. If $y \in V(H)$ is a vertex of degree 1, then let $e \in E(H)$ be the incident edge. Otherwise, all vertices of H have degree either 0 or 2. If this is the case, let $y \in V(H)$ be a vertex of degree 2, and let $e \in E(H)$ be an edge incident with y. By the edge-minimality of H, there is, in either case, an embedding

$$\pi: \quad V(H) \longrightarrow V(G)$$

of $H - e$ into G. Let $\pi(y)$ be denoted by x. <u>The bijection π and the vertex x are considered fixed throughout the proof</u>. At a relatively early stage in the proof (prop. 10.5), we shall dispose of the case in which H has a vertex of degree 1.

Henceforth, all vertices denoted in this proof by a letter are vertices of G.

We define an <u>alternating chain</u> from x_0 to v to be any finite sequence of at least 2 distinct vertices $x_0, x_1, \ldots, x_m$ of $V(G)$, with $x_m = v$, such that

(10.2) $\quad x_i \in S(x_{i-1}) \qquad$ for $i = 1, 2, \ldots, m$;

(10.3) $\quad$ If $x_i = x_j$ and $i < j$, then either $x_i = x_{i+1} = \ldots = x_j$, or both $x_0 = x_1 = \ldots = x_i$ and $x_j = \ldots = x_m = x_0$;

and (10.4) $\quad x_i \notin M(x_j) \qquad$ for $0 \leq i < j \leq m$.

Note that (10.2) is equivalent to $x_{i-1} \in P(x_i)$, and (10.4) is equivalent to $x_j \notin M(x_i)$. If $x_0 = v$ in an alternating chain from x_0 to v, we say that the chain is <u>closed</u>. The proof of Theorem 10.1 will rest upon the observation that for a closed alternating chain $x_0, x_1, \ldots, x_m$, with $x = x_0 = x_m$, $(x_0\ x_1\ \ldots\ x_{m-1})\pi$ is an embedding of H into G.

Define, for each integer $t \geq 1$, the set $D_t(x)$ to be the set of all vertices z such that for any $t-1$ vertices $w_1, w_2, \ldots, w_{t-1} \in V(G) - M(x,z)$, there is an alternating chain from x to z containing no vertex of $M(w_1, \ldots, w_{t-1})$. Define $D_0(x) = V(G)$. Thus, we have

(10.5) $D_0(x) \supseteq D_1(x) \supseteq D_2(x) \supseteq \ldots \supseteq D_t(x) \supseteq \ldots \supseteq S(x)$.

We first prove 15 propositions. Then we break the proof into six cases, and use the propositions and two key lemmas about alternating chains to show that in each case there is a permutation $\alpha: V(G) \longrightarrow V(G)$ such that $\alpha\pi$ embeds H into G.

Recall that in section 1 we defined, for $x_1, x_2, \ldots, x_n \in X$, where $x_i = x_j$ and $i < j$ imply $x_i = x_{i+1} = \cdots = x_j$, the symbol $(x_1\ x_2\ \ldots\ x_n)'$ to be the permutation obtained by suppressing multiple successive occurrences of the same member of X in $x_1, x_2, \ldots, x_n$. Thus, if $v_0, v_1, v_2 \in V(G)$ are distinct and if $v_2 = v_3$, then

$$(v_0\ v_1\ v_2\ v_3)' = (v_0\ v_1\ v_2).$$

<u>Prop. 10.2</u> The number $|P(v)|$ of predecessors of a vertex $v \in V(G)$ is at least

$$\frac{p}{3} + \max(2k, 3p^{1/3}) - 2.$$

<u>Proof</u>: A vertex v' is not a predecessor of v if there is a vertex $u \in V(G)$, either equal or adjacent in G^c to v, such that $v' \in M(u)$. Since $|M(u)| \leq 2$, (10.1) implies that there are at most

$$\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$$

non-predecessors of v. Prop. 10.2 follows.

<u>Prop. 10.3</u> The number $|S(v)|$ of successors of a vertex v is at least

$$\frac{p}{3} + \max(2k, 3p^{1/3}) - 2.$$

<u>Proof</u>: The non-successors of v are the vertices which are either equal or adjacent in G^c to a vertex of $M(v)$. For each $u \in M(v)$, the number of vertices either equal or not adjacent to u is at most $\frac{p}{3} - \max(k, \frac{3}{2}p^{1/3}) + 1$. Since $|M(v)| \leq 2$, there are at most $\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$ non-successors of v. Prop. 10.3 follows.

<u>Prop. 10.4</u> If $z \in D_t(x)$ and $t \geq 1$, then $z \notin P(x) + x$.

<u>Proof</u>: If Prop. 10.4 is false, then there is a closed alternating chain $x = x_0, x_1, \ldots, z, x$, and so $(x_0\ x_1\ \ldots\ z)'\pi$ embeds H into G. This is the conclusion of Theorem 10.1, which we have assumed to be false.

Prop. 10.5 Every vertex of H has degree either 0 or 2. In particular, $|M(x)| = 2$, and if $|M(v)| = 0$ for some $v \in V(G)$, then $v \in P(x)$.

Proof: If $|M(x)| = 1$, then let $M(x) = x'$. The successors of x are the vertices adjacent in G to x'. Thus, by (10.1), $|S(x)| > \frac{2p}{3}$, and since, by Prop. 10.2, there are more than $\frac{p}{3}$ predecessors of x, there is a vertex $x_1 \in S(x) \cap P(x)$. Then $(x\ x_1)\pi$ embeds H into G, contrary to the assumption that H is a graph for which the theorem is false. Hence, $|M(x)|$ is not 1, and by the original choice of x, every vertex of H has degree 0 or 2. The final statement of the proposition follows because, by the definition of successors, if $|M(v)| = 0$, then $S(v) = V(G)$.

Definitions If $z^* \in D_1(x) - D_t(x)$, for some $t \geq 2$, define $\mathcal{C}(z^*)$ to be the set of all alternating chains from x to z^*. Since $z^* \notin D_t(x)$, then by definition of $D_t(x)$, there is a set $A(z^*) = \{w_1, \ldots, w_s\} \subseteq V(G) - M(x, z^*)$, with minimum possible integer $s \leq t - 1$, such that every chain in $\mathcal{C}(z^*)$ has a vertex in $M(A(z^*))$. Of course, $A(z^*)$ is not necessarily uniquely determined. However, we shall consider the set $A(z^*)$ to be fixed, for each $z^* \in D_1(x) - D_t(x)$. We have

$$|A(z^*)| \leq t-1,$$

and we have

$$|M(A(z^*))| \leq 2t-2.$$

Prop. 10.6 Let $t \geq 2$ be an integer. If

(10.6) $z \in D_t(x)$

and if

(10.7) $z^* \in S(z) - D_t(x)$,

then one of the following three statements holds:

(10.8) $z \notin M(A(z^*)) \cap D_{t+1}(x)$ and $z^* \in D_{t-1}(x)$;

(10.9) $z \in M(A(z^*))$;

(10.10) $z^* \in M(x)$.

If (10.10) is false, then also,

(10.11) $z^* \notin P(x) + x$.

Proof: Let $z, z^* \in V(G)$ satisfy (10.6) and (10.7). First, we claim that either (10.10) holds, or $\mathcal{C}(z^*)$ is not empty. By (10.6) and $t \geq 2$, there is a chain C from x to z avoiding $M(z^*)$. If C passes through z^*, then $\mathcal{C}(z^*)$ is not empty. Otherwise, we extend the chain C by adding z^* at the end and we denote the resulting sequence by C^*. Since $z^* \in S(z)$, C^* is an alternating chain, provided some vertex in $M(z^*)$ does not already occur in C^*. By our choice of C, unless (10.10) holds, this condition is satisfied. This justifies the claim.

Henceforth, we assume that (10.10) is false and thus that $\mathcal{C}(z^*)$ is not empty. Therefore, $A(z^*)$ exists, and $z^* \in D_1(x)$, whence (10.11) follows, by Prop. 10.4.

By (10.6) and (10.7),

(10.12) $\quad z \in P(z^*) \frown D_t(x)$.

Suppose by way of contradiction that (10.9) is false and

(10.13) $\quad z \in D_{t+1}(x)$.

Then either there is an alternating chain C from x to z that misses $M(A(z^*) + z^*)$, or by definition of $D_{t+1}(x)$, the sets $A(z^*) + z^*$ and $M(x,z)$ intersect.

We quickly dispose of the latter possibility. From (10.12), $z \in P(z^*)$, and hence, $z^* \notin M(z)$. Since (10.10) is assumed to be false, $z^* \notin M(x)$. Since (10.9) is assumed to be false, $M(z)$ cannot intersect $A(z^*)$. Finally, the definition of $A(z^*)$ assures us that $M(x)$ and $A(z^*)$ do not overlap.

Thus, we can assume that there is an alternating chain C from x to z that misses $M(A(z^*) + z^*)$. If z^* does not occur in C, let C^* denote the sequence obtained by appending z^* to the end of the sequence C. Since C misses $M(z^*)$, $C^* \in \mathcal{C}(z^*)$, unless z^* occurs in C. But if z^* occurs in C, let C^* instead denote the subsequence of C terminating at z^*. Since C misses $M(A(z^*))$ and since, by definition of $A(z^*)$, z^* also misses $M(A(z^*))$, so does $C^* \in \mathcal{C}(z^*)$, contrary to the definition of $A(z^*)$. This contradiction shows that (10.13) is false, and thus the first part of (10.8) holds.

Next, supposing that (10.9) and (10.10) are false, we shall prove the last part of (10.8). We proceed by induction on t.

As a basis for induction, we note that the fact that $\mathcal{C}(z^*)$ is nonempty implies that $z^* \in D_1(x)$, by definition of $D_1(x)$. Therefore, the last part of (10.8) holds when $t = 2$.

Suppose that Prop. 10.6 is true for integers less than t, where $t \geq 3$. Suppose, contrary to (10.8), that

(10.14) $\quad z^* \notin D_{t-1}(x)$.

Thus, (10.5) and (10.6) imply

(10.15) $\quad z \in D_{t-1}(x)$,

and (10.14) and (10.7) imply

(10.16) $\quad z^* \in S(x) - D_{t-1}(x)$.

Note that (10.15) and (10.16) are simply (10.6) and (10.7) with $t - 1$ in place of t. Hence, by the induction hypothesis, since (10.9) and (10.10) are false, we must have $z \notin D_t(x)$. But this contradicts (10.6) itself. Therefore, (10.8) is proved. This proves Prop. 10.6.

We define a succession to be an ordered pair (u,v) of vertices such that $v \in S(u)$.

<u>Prop. 10.7</u> There is an integer $t \geq 2$ such that the number of successions (u,v) with $u \in D_t(x)$, $v \notin D_t(x)$ is at most $p^{4/3} + \frac{2}{3}p$.

<u>Proof</u>: We have the following three upper bounds on different types of successions, for $t \geq 2$.

(10.17) The number of successions (u,v), with $u \in D_t(x)$, $v \in M(x)$ is at most $2|D_t(x)|$;

(10.18) The number of successions (u,v) with $v \notin D_{t-1}(x) \cup M(x)$ and $u \in D_t(x) \cap P(v)$, with $|D_t(x) \cap P(v)| \leq 2t-2$, is at most $(p - |P(x)| - 1 - |D_{t-1}(x)|)(2t-2)$;

(10.19) The number of successions (u,v), with $u \in D_t(x)$, $v \notin D_t(x)$ and $|D_t(x) \cap P(v)| > 2t-2$ is at most

$$(|D_{t-1}(x) - D_t(x)|)(2t-2+|D_t(x) - D_{t+1}(x)|).$$

Statement (10.17) holds because by Prop. 10.5,

$$|M(x)| = 2.$$

We obtain (10.18) by using (10.11) of Prop. 10.6, which asserts that $v \notin P(x) + x$. Next, we justify the bound of (10.19).

Suppose that for a given vertex $v \notin D_t(x) \cup M(x)$, there are more than $2t-2$ successions of the form (u,v), where $u \in D_t(x)$. Thus, we have excluded successions counted in (10.17) and (10.18). By Prop. 10.6 with $u = z$ and $v = z^*$, there is a set $M(A(v))$ of at most $2t-2$ vertices in $V(G)$ such that (10.8), (10.9), or (10.10) of Prop. 10.6 holds. We cannot have (10.10), since $v \in M(x)$ has been excluded. Note that if for each u, condition (10.9) holds, then there are at most $2t-2$ values of u, another case already excluded. Hence, there is a value of u, say $u = u_0$, such that $u_0 \notin M(A(v))$. Then we have (10.8), whence $u_0 \in D_t(x) - D_{t+1}(x)$, and $v \in D_{t-1}(x)$. Therefore, in general, since

$$u \in M(A(v)) \cup (D_t(x) - D_{t+1}(x))$$

and

$$v \in D_{t-1}(x) - D_t(x)$$

for all successions (u,v) not counted in (10.17) or (10.18), the bound of (10.19) holds.

We write

$$(10.20) \quad a_t = |D_t(x) - D_{t+1}(x)| .$$

The total number of successions (u,v) with $u \in D_t(x)$, $v \notin D_t(x)$ is, by (10.17), (10.18), and (10.19), at most

$$(10.21)\quad a_{t-1}(2t-2+a_t) + (p-|P(x)|-1-|D_{t-1}(x)|)(2t-2) + 2|D_t(x)|$$
$$= a_{t-1}a_t + (p-|P(x)|-1-|D_t(x)|)(2t-2) + 2|D_t(x)|.$$

Let

$$(10.22)\quad b = p^{4/3}$$

and let

$$(10.23)\quad c = 2p/3.$$

Suppose, by way of contradiction, that for all t satisfying $2 \leq t \leq b/c$,

$$(10.24)\quad a_t a_{t-1} \geq b - ct.$$

Since

$$0 \leq (\sqrt{a_t} - \sqrt{a_{t-1}})^2 = a_t - 2\sqrt{a_t}\sqrt{a_{t-1}} + a_{t-1},$$

we have from this and (10.24),

$$(10.25)\quad \sqrt{(b-ct)} \leq \sqrt{(a_t a_{t-1})} \leq \tfrac{1}{2}(a_t + a_{t-1}).$$

Summing (10.25) from $t=2$ to $n=[b/c]$, we get

$$(10.26)\quad \Sigma_{t=2}^{n} \sqrt{(b-ct)} \leq -\tfrac{1}{2}a_1 - \tfrac{1}{2}a_n + \Sigma_{t=2}^{n} a_t$$
$$< \Sigma_{t=2}^{n} a_t.$$

By the Fundamental Theorem of Calculus, and (10.26),

$$(10.27)\quad \frac{2b^{3/2}}{3c} = \int_0^{b/c} \sqrt{(b-cx)}\,dx$$
$$= \int_0^2 \sqrt{(b-cx)}\,dx + \int_2^{b/c} \sqrt{(b-cx)}\,dx$$
$$\leq \int_0^2 \sqrt{b}\,dx + \Sigma_{t=2}^{n} \sqrt{(b-ct)}$$
$$< 2\sqrt{b} + \Sigma_{t=2}^{n} a_t.$$

We combine (10.22), (10.23), (10.27), Prop. 10.4, and Prop. 10.2 to obtain

$$\begin{aligned} p &= \frac{2b^{3/2}}{3c} \\ &< 2p^{2/3} + \Sigma_{t=2}^{n} a_t \\ &= 2p^{2/3} + |D_2(x) - D_{n+1}(x)| \\ &\leq 2p^{2/3} + p - |(P(x) + x)|. \\ &< 2p^{2/3} + \frac{2p}{3} - 14, \end{aligned}$$

which is clearly false for all p. Hence, there is a t such that (10.24) is false.

Fix t throughout the rest of the proof so that

(10.28) $a_t a_{t-1} < b - ct.$

Throughout the rest of the proof, let

(10.29) $D_t(x) = D(x),$

for this value of t satisfying (10.28).

Thus, by (10.28), (10.29), (10.21), (10.5) and Props. 10.4, 10.3, and 10.2, and by (10.22) and (10.23),

$$\begin{aligned} b - ct + (p - |P(x)| - 1 - |D(x)|)(2t - 2) + 2|D(x)| \\ < b - ct + (2t - 2)\frac{p}{3} + 2\left(\frac{2p}{3}\right) \\ = p^{4/3} + \frac{2p}{3}. \end{aligned}$$

This proves Prop. 10.7.

Prop. 10.8 For any $u_0 \in D(x)$, the number of successions of the form (u,v), with $u,v \in D(x) - u_0$, is at least

$$(|D(x)| - 1)(\frac{p}{3} + \max(2k, 3p^{1/3}) - 3) - p^{4/3} - \frac{2}{3}p.$$

Proof: By Prop. 10.3, the number of successions of the form (u,v), with $u \in D(x) - u_0$ and $v \neq u_0$ is at least

$$|D(x) - u_0|(\frac{p}{3} + \max(2k, 3p^{1/3}) - 2 - |\{u_0\}|).$$

By Prop. 10.7, at most $p^{4/3} + \frac{2p}{3}$ of these are not of the form with $v \in D(x)$. This implies the proposition.

Prop. 10.9 There are distinct vertices u_0, v_0 in $D(x)$ such that

$$(10.30) \quad |P(u_0) \cap D(x)| \geq |P(v_0) \cap D(x)| > \frac{p}{3} - 5.$$

Proof: Let $u_0 \in D(x)$ be a vertex having the most predecessors in $D(x)$. Let v_0 denote a vertex in $D(x) - u_0$ having the most predecessors in $D(x) - u_0$. Clearly, the first inequality of (10.30) holds. Note that $|P(v_0) \cap D(x)|$ is at least the average number of successions per vertex in $D(x) - u_0$, whence, by Prop. 10.8,

$$(10.31) \quad |P(v_0) \cap D(x)| \geq \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 - \frac{p^{4/3} + 2p/3}{|D(x)| - 1}$$

By Prop. 10.3, and since $S(x) \subseteq D(x)$,

$$(10.32)\quad |D(x)| - 1 \geq |S(x)| - 1$$
$$\geq \frac{p}{3} + \max(2k, 3p^{1/3}) - 3$$
$$> \frac{p}{3}.$$

By (10.31) and (10.32),

$$|P(v_0) \cap D(x)| > \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 - 3p^{1/3} - 2$$
$$\geq \frac{p}{3} - 5,$$

and hence, (10.30) holds.

Remarks: Vertices u_0 and v_0 satisfying Prop. 10.9 are chosen, and will remain fixed throughout the rest of the proof of Theorem 10.1.

Also, for the remainder of the proof, we shall use (10.1) and Props. 10.2 and 10.3 in their weaker form, without the term involving $p^{1/3}$. Thus, the inequalities of (10.1) and Props. 10.1 and 10.2 will be replaced by

$$\Delta(G^c) \leq \frac{p}{3} - k,$$
$$|P(v)| \geq \frac{p}{3} + 2k - 2,$$
$$|S(v)| \geq \frac{p}{3} + 2k - 2,$$

respectively.

Prop. 10.10 We have both

$$|S(x) \cap P(u_0)| > 4k - 8,$$

and

$$|S(x) \cap P(v_0)| > 4k - 8,$$

where u_0 and v_0 are the fixed vertices of Prop. 10.9.

<u>Proof</u>: Since the proofs are identical for u_0 and v_0, we shall only state the proof for v_0. By Prop. 10.2,

$$(10.33)\quad |V(G) - P(x)| \leq \frac{2p}{3} - 2k + 2.$$

Since, by Prop. 10.4,

$$D(x) \subseteq V(G) - P(x) - x,$$

(10.33) gives

$$(10.34)\quad |D(x)| \leq \frac{2p}{3} - 2k + 1.$$

By (10.34), Prop. 10.9, Prop. 10.3, and $S(x) \subseteq D(x)$, we have

$$\begin{aligned} |S(x) \cap P(v_0) \cap D(x)| &\geq |S(x) \cap D(x)| + |P(v_0) \cap D(x)| - |D(x)| \\ &> (\tfrac{p}{3} + 2k - 2) + (\tfrac{p}{3} - 5) - (\tfrac{2p}{3} - 2k + 1) \\ &\geq 4k - 8. \end{aligned}$$

<u>Prop. 10.11</u> Suppose that v_1 and v_2 satisfy

$$(10.35)\quad v_1 \in S(v_0) - M(x)$$

and

$$(10.36)\quad v_2 \in S(v_1) - M(x, v_0).$$

Then $v_2 \notin P(x) + x$, and either $v_0 = v_2$ or

$$|S(v_2) - (P(x) + x)| \geq \frac{p}{3} + 2k - 8.$$

A similar statement holds when v_0, v_1, v_2 are replaced by u_0, u_1, u_2, respectively.

<u>Proof</u>: By Prop. 10.10, since $k > 4$, there is a vertex

$$x_1 \in S(x) \cap P(v_0) - M(v_1, v_2) - \{v_0, v_1, v_2\},$$

and $x_1 \in S(x) \cap P(v_0)$ guarantees that $x_1 \notin M(x, v_0)$ and $v_0 \in S(x_1)$. We claim that x, x_1, v_0, v_1, v_2 is an alternating chain, or that $v_0 = v_2$. To see this, observe first that (10.2) holds for this sequence. Suppose next, that (10.3) fails for this sequence. By the choice of x_1, $x_1 \notin \{x, v_0, v_1, v_2\}$. Thus, for (10.3) to fail, either $x \in \{v_0, v_1\}$ or $v_0 = v_2$. If $x \in \{v_0, v_1\}$, then either x, x_1, v_0 or x, x_1, v_0, v_1 is a closed alternating chain, whence $(x\ x_1\ v_0)\pi$ or $(x\ x_1\ v_0\ v_1)\pi$, respectively, embeds H into G, contrary to the assumption that H is not a subgraph of G. If $v_0 = v_2$, then the proposition follows immediately, since $v_0 \notin P(x) + x$. Suppose, finally, that x, x_1, v_0, v_1, v_2 is not an alternating chain because (10.4) fails. Thus, there are vertices

$$y_1, y_2 \in \{x, x_1, v_0, v_1, v_2\}$$

such that $y_0 \in M(y_2)$. By definition of successors, y_1 and y_2 cannot be consecutive vertices of x, x_1, v_0, v_1, v_2. Since $v_0 \in D(x)$, we have $v_0 \notin M(x)$. By the definitions of v_1, v_2, and x_1, we exclude $v_1 \in M(x)$, $v_2 \in M(x)$, $v_1 \in M(x_1)$, $v_2 \in M(x_1)$ and $v_2 \in M(v_0)$. Thus, if $v_0 \neq v_2$, then (10.2), (10.3), and (10.4) hold, and so x, x_1, v_0, v_1, v_2 is an alternating chain.

If $v_2 \in P(x) + x$, then we are done, for $(x\ x_1\ v_0\ v_1\ v_2)\pi$ would be an embedding of H into G.

If there exists a vertex

$$v_3 \in S(v_2) \cap P(x) - M(v_0, v_1, x_1),$$

then $x, x_1, v_0, v_1, v_2, v_3, x$ is a closed alternating chain, and we are done, for $(x\ x_1\ v_0\ v_1\ v_2\ v_3)\pi$ embeds H into G. Otherwise, all members of $S(v_2) \cap (P(x) + x)$ lie in $M(v_0, v_1, x_1)$, a set of at most 6 members. The number of successors of v_2 outside of $P(x) + x$ is therefore, by Prop. 10.3, at least $\frac{p}{3} + 2k - 2 - 6$. This proves Prop. 10.11.

<u>Prop. 10.12</u> For any two vertices u and v in V(G), the number of predecessors of u adjacent in G to v is at least $3k - 3$.

<u>Proof</u>: By Prop. 10.2,

$$|P(u)| \geq \frac{p}{3} + 2k - 2.$$

Since

$$\Delta(G^c) \leq \frac{p}{3} - k,$$

at most $\frac{p}{3} - k + 1$ vertices of P(u) are adjacent in G^c to v (one is equal). This leaves at least

$$\left(\frac{p}{3} + 2k - 2\right) - \left(\frac{p}{3} - k + 1\right)$$

predecessors of u adjacent to v.

Prop. 10.13 For any two vertices u and v in V(G), the number of successors of u that are adjacent in G to v is at least $3k - 3$.

Proof: Use the proof of Prop. 10.12, with Prop. 10.2 replaced by Prop. 10.3.

Prop. 10.14 Suppose that v_1 satisfies (10.35). Let y_1, z_2 be two vertices of G such that y_1 is adjacent in G to all successors of v_1. Then the number of successors of v_1 outside $P(x) + x$ that are adjacent in G to z_2 and y_1 is at least $3k - 7$.

Proof: By the first conclusion of Prop. 10.11, at most 4 successors v_2 of v_1 lie in $P(x) + x$ (namely, $M(x, v_0)$), whence, by Prop. 10.3, at least

$$|S(v_1) - P(x) - x| \geq \frac{p}{3} + 2k - 6$$

successors of v_1 lie outside $P(x) + x$. At most $\frac{p}{3} - k + 1$ of these are not adjacent in G to z_2, by (10.1). All are adjacent to y_1, by hypothesis. This leaves at least $3k - 7$ vertices.

Prop. 10.15 If v_2 satisfies condition (10.36) of Prop. 10.11 and if $v_2 \neq v_0$, then

$$|S(v_2) \cap P(v_0) - (P(x) + x)| > 4k - 14.$$

Proof: By Prop. 10.11,

(10.37) $|S(v_2) - (P(x) + x)| \geq \frac{p}{3} + 2k - 8.$

By Prop. 10.4 and 10.9,

(10.38) $|P(v_0) - (P(x) + x)| \geq |P(v_0) \cap D(x)|$

$> \frac{p}{3} - 5.$

By (10.37), (10.38), and Prop. 10.2,

$$|S(v_2) \cap P(v_0) - (P(x) + x)| \geq |S(v_2) - (P(x) + x)|$$
$$+ |P(v_0) - (P(x) + x)| - |V(G) - (P(x)+x)|$$
$$> (\tfrac{p}{3} + 2k - 8) + (\tfrac{p}{3} - 5) - p + \tfrac{p}{3} + 2k - 1$$
$$= 4k - 14.$$

Prop. 10.16 For appropriate vertices $x_1 \in S(x)$, $u_1 \in S(u_0) - M(x)$, $v_1 \in S(v_0) - M(x)$, and $z_1 \in V(G)$, one of the following six cases holds:

(10.39) $x_1 \in M(v_1)$;

(10.40) $x_1 \in M(u_1)$;

(10.41) $v_1 \in M(u_1)$;

(10.42) $M(z_1) = \{x_1, v_1\}$;

(10.43) $M(z_1) = \{x_1, u_1\}$;

(10.44) $M(z_1) = \{u_1, v_1\}$.

Proof: Let

$$X = (S(x) \cap (S(u_0) - M(x))) \cup (S(x) \cap (S(v_0) - M(x)))$$
$$\cup ((S(u_0) - M(x)) \cap (S(v_0) - M(x))),$$

and let

$$X' = S(x) \cup S(u_0) \cup S(v_0) - M(x).$$

Note that the definitions of $S(x)$, X, and X' imply that these sets are disjoint from $M(x)$. If (10.39), (10.40), and (10.41) are false, then for any vertex $z \in X$,

$$M(z) \subseteq V(G) - X'.$$

If also, (10.42), (10.43), and (10.44) are false, then the sets $M(z)$, where z runs over X, are disjoint sets of 2 elements contained in $V(G) - X'$. Hence,

$$(10.45) \qquad |V(G) - X'| \geq 2|X|.$$

By Prop. 10.3, we have

$$\begin{aligned} |S(x) - X| + |X| &\geq |S(x)| \\ &\geq \frac{p}{3} + 2k - 2 \\ |S(u_0) - M(x) - X| + |X| &\geq |S(u_0) - M(x)| \\ &\geq \frac{p}{3} + 2k - 4, \\ |S(v_0) - M(x) - X| + |X| &\geq |S(v_0) - M(x)| \\ &\geq \frac{p}{3} + 2k - 4, \end{aligned}$$

whence,

$$(10.46) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X| + 3|X| \geq p + 6k - 10.$$

We also have

$$(10.47) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X| + |X| = |X'|.$$

We combine (10.46) and (10.47) to obtain

$$p + 6k - 10 \leq |X'| + 2|X|,$$

whence

$$p - |X'| + 6k - 10 \leq 2|X|,$$

and so, since $6k > 10$,

$$|V(G) - X'| < 2|X|,$$

in contradiction with (10.45). Prop. 10.16 follows.

In the two lemmas below, we define for vertices of G,

$$X = \{x_0, x_1, \ldots, x_n\};$$
$$V = \{v_0, v_1, \ldots, v_m\};$$
$$\alpha = (x_0\ x_1\ \ldots\ x_n);$$
$$\beta = (v_0\ v_1\ \ldots\ v_m).$$

Let $G + \{x_i, v_j\}$ denote the graph obtained from G by adding to E(G) the edge $\{x_i, v_j\}$, where $x_i, v_j \in V(G)$.

<u>Lemma 10.17</u> Let $x_0, x_1, \ldots, x_n, x_0$ be a closed alternating chain in $G + \{x_2, v_1\}$, and let $v_0, v_1, \ldots, v_m, v_0$ be a closed alternating chain in $G + \{x_1, v_2\}$. If

(10.48) $x_1 \in M(v_1)$,

(10.49) $x \in X$,

(10.50) $\{x_2, v_2\} \in E(G)$,

and if

(10.51) $V \cap (M(X) \cup X) = v_1$,

then $\beta\alpha\pi$ is an embedding of H into G.

<u>Proof</u>: By (10.48), there is an edge e' of H mapped by π to v_1, x_1. Recall that e is the edge of H not mapped into E(G) by π and that $x \in \pi(e)$.

By hypothesis and by (10.49), $\alpha\pi$ embeds H into $G + \{x_2, v_1\}$ and maps e' to $\{x_2, v_1\}$. Also, by hypothesis,

$\beta\pi$ embeds $H - e$ into $G + \{x_1, v_2\}$, and maps e' to $\{x_1, v_2\}$. By (10.51), e' is the only edge of H affected by both α and β. Hence, $\beta\alpha\pi$ embeds $H - e'$ into G, and since

$$\beta\alpha\pi(e) = \{x_2, v_2\} \in E(G),$$

by (10.50), $\beta\alpha\pi$ embeds H into G.

Lemma 10.18 Let $x_0, x_1, \ldots, x_n, x_0$ be a closed alternating chain in $G + \{x_2, z_1\}$, and let $v_0, v_1, \ldots, v_m, v_0$ be a closed alternating chain in $G + \{v_2, z_1\}$. Also, let $Z = \{z_1, z_2\}$, and let $\gamma = (z_1\ z_2)$. If

(10.52) $M(z_1) = \{v_1, x_1\}$,

(10.53) $x \in X$,

(10.54) $\{v_2, z_2\}, \{x_2, z_2\} \in E(G)$,

(10.55) $(V \cup Z) \cap (M(X) \cup X) = z_1$,

(10.56) $(X \cup Z) \cap (M(V) \cup V) = z_1$,

and if

(10.57) $z_2 \in P(z_1)$,

then $\gamma\beta\alpha\pi$ embeds H into G.

Proof: By (10.52), there are edges e_1, e_2, respectively, mapped by π to $\{x_1, z_1\}$ and $\{z_1, v_1\}$. Recall that e is the only edge of H not mapped into E(G) by π, and that $x \in \pi(e)$.

By the first hypothesis, and by (10.53), $\alpha\pi$ embeds H into $G + \{x_2, z_1\}$, with e_1 mapped to $\{x_2, z_1\}$. Also, by hypothesis, $\beta\pi$ embeds $H - e$ into $G + \{v_2, z_1\}$. By (10.55) and (10.56), no edge of H is affected by both α and β. Therefore, $\beta\alpha\pi$ embeds H into $G + \{x_2, z_1\} + \{v_2, z_1\}$, with e_1, e_2 mapped to $\{x_2, z_1\}, \{v_2, z_1\}$, respectively. By (10.55) and (10.56), e_1, respectively e_2, is the only edge affected by both α and γ, respectively β and γ. Hence, (10.57) ensures that $\gamma\beta\alpha\pi$ embeds $H - \{e_1, e_2\}$ into G, and since $\gamma\beta\alpha\pi$ maps e_1 and e_2 to $\{x_2, z_2\}$ and $\{v_2, z_2\}$, (10.54) ensures that $\gamma\beta\alpha\pi$ embeds H into G.

Remark: In the six cases of Prop. 10.16 which we consider below, we shall verify the hypotheses of either Lemma 10.17 or Lemma 10.18, whence we conclude that H is a subgraph of G. We shall construct the desired alternating chains one vertex at a time. Each time another vertex is chosen, we take care to ensure that (10.51) or both (10.55) and (10.56) hold, although we shall not say so explicitly. As chains are constructed, we select vertices which satisfy (10.2), (10.3), and (10.4). Again, we do not refer to these three conditions explicitly. The other conditions of the lemmas will be verified explicitly in each of the six cases.

Suppose (10.39) holds. We shall apply Lemma 10.17 to show that $(v_0\ v_1\ v_2\ v_3)'\ (x\ x_1\ x_2)\pi$ embeds H into G, for vertices v_2, v_3, x_2 defined below. We must verify the hypotheses of the lemma. If $x_1 \in M(v_0)$, then we pick v_1 to equal v_0. This is in compliance with (10.39). Already by (10.39), we have (10.48), and clearly we have (10.49). By Prop. 10.5, there exists $w_1 \in V(G)$ such that

$$M(x_1) = \{v_1, w_1\}.$$

Define the set

$$T_1 = \{v_0, v_1, w_1, x, x_1\} \cup M(v_0, v_1, x, x_1).$$

Since x_1, v_1, and w_1 are counted twice,

$$|T_1| \leq 10.$$

By Prop. 10.12, since

$$3k - 3 > 10 \geq |T_1|,$$

a vertex $x_2 \in P(x) - T_1$ exists adjacent in G to w_1. Thus x_2 is a successor of x_1 in $G + \{x_2, v_1\}$, whence x, x_1, x_2, x is the desired alternating closed chain in $G + \{x_2, v_1\}$. Let

$$T_2 = T_1 \cup (M(x_2) + x_2).$$

Hence,

$$|T_2| \leq 13.$$

By Prop. 10.13, since

$$3k - 3 > 13 \geq |T_2|,$$

a vertex $v_2 \in S(v_1) - T_2$ exists adjacent in G to x_2, in accordance with (10.50). Let

$$T_3 = T_2 \cup (M(v_2) + v_2).$$

Thus,

$$|T_3| \le 16.$$

If $v_2 \in P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, by Prop. 10.15, since

$$4k - 14 > 16 \ge |T_3|,$$

there is a vertex

$$v_3 \in S(v_2) \wedge P(v_0) - (P(x) + x) - T_3.$$

Thus, v_0, v_1, v_2, v_3, v_0 is a closed alternating chain in $G = G + \{v_2, x_1\}$, as desired. We have chosen the vertices v_2, v_3, x_2 so that (10.51) holds, whence Lemma 10.17 may be applied.

<u>Suppose (10.40) holds</u>. This case proceeds as does the previous case with (10.39), but with v replaced by u, and so we omit the proof.

<u>Suppose (10.41) holds</u>. We shall apply Lemma 10.17 to show that $(v_0\ v_1\ v_2\ v_3)'\ (u_0\ u_1\ x_2\ x_3\ x_4)'\pi$ embeds H into G, for vertices v_2, v_3, x_2, x_3, x_4 defined below. We must verify the hypotheses of the lemma. Let

$$x_3 = x.$$

Thus, (10.49) holds. We shall apply Lemma 10.17 with u_1 corresponding to x_1 of the lemma, whence by (10.41), (10.48) holds. If $v_1 \in M(u_0)$, then let $u_1 = u_0$ in this argument. Define the set

$$T_1 = \{u_0, v_0, x\} \cup M(u_0, u_1, v_0, v_1, x).$$

By (10.41), $\{u_1, v_1\} \subseteq T_1$, and so by Prop. 10.10, since

$$4k - 8 > 13 \geq |T_1|,$$

there is a vertex $x_4 \in S(x) \cap P(u_0) - T_1$. By Prop. 10.5, there exists $w_1 \in V(G)$ such that

$$M(u_1) = \{v_1, w_1\}.$$

Define

$$T_2 = T_1 \cup (M(x_4) + x_4).$$

By Prop. 10.12, since

$$3k - 3 > 16 \geq |T_2|,$$

there is a vertex $x_2 \in P(x) - T_2$ adjacent in G to w_1. Observe that $u_0, u_1, x_2, x_3, x_4, u_0$ is a closed alternating chain in $G + \{v_1, x_2\}$. Thus, we have the first desired chain of Lemma 10.17. Define

$$T_3 = T_2 \cup (M(x_2) + x_2).$$

By Prop. 10.13, since

$$3k - 3 > 19 \geq |T_3|,$$

a vertex $v_2 \in S(v_1) - T_3$ exists adjacent in G to x_2, in compliance with (10.50). If $v_2 \in P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, by Prop. 10.15, since

$$4k - 14 \geq 19 \geq |T_3|,$$

a vertex

$$v_3 \in S(v_2) \cap P(v_0) - T_3 - (P(x) + x)$$

exists. Since $v_3 \in S(v_2)$, we have $v_3 \notin M(v_2)$. Thus, v_0, v_1, v_2, v_3, v_0 is a closed alternating chain in $G + \{x_1, v_2\}$ that satisfies the conditions of the first chain lemma. By the lemma, H is a subgraph of G.

<u>Suppose (10.42) holds</u>. We shall apply Lemma 10.18 to show that $(z_1\ z_2)(v_0\ v_1\ v_2\ v_3)'(x\ x_1\ x_2)\pi$ embeds H into G. Thus, we already have (10.52) and (10.53).

Without loss of generality, we may assume in this case that $x_1 \notin M(v_0)$, for otherwise, we would use the argument associated with (10.39). By Prop. 10.5, and be (10.42), there is a vertex $w_1 \in V(G)$ such that

$$M(x_1) = \{w_1, z_1\}.$$

If the vertex of H mapped to x_1 lies in a triangular component of H, then $w_1 = v_1$ and the triangle is mapped to the vertices x_1, v_1, z_1, and hence (10.39) holds. Hence, without loss of generality, we may assume that w_1, x_1, z_1, v_1 are distinct vertices in the image of a path of H. Thus, for the vertices of V, X, and Z already selected, (10.55) and (10.56) hold. Define

$$T_1 = \{v_0, w_1, x\} \cup M(v_0, v_1, x, z_1).$$

Note that

$$v_1, x_1, z_1 \in M(v_1, z_1) \subseteq T_1,$$

and that

$$|T_1| \leq 11.$$

By Prop. 10.12, since

$$3k - 3 > 11 \geq |T_1|,$$

a vertex $x_2 \in P(x) - T_1$ exists adjacent in G to w_1. Thus,

x, x_1, x_2, x is a closed alternating chain in $G + \{x_2, z_1\}$, in compliance with Lemma 10.18. Define

$$T_2 = T_1 \cup (M(x_2) + x_2).$$

By Prop. 10.12, since

$$3k - 3 > 14 \geq |T_2|,$$

a vertex $z_2 \in P(z_1) - T_2$ exists adjacent in G to x_2, in accordance with (10.57) and the second part of (10.54). Define

$$T_3 = T_2 \cup (M(z_2) + z_2).$$

By Prop. 10.5 and (10.42), there exists $y_1 \in V(G)$ such that

$$M(v_1) = \{y_1, z_1\}.$$

Since $y_1 \in M(v_1)$, y_1 is adjacent in G to all successors of $v_1 \in S(v_0) - M(x)$. By Prop. 10.14, since

$$3k - 7 > 17 \geq |T_3|,$$

a vertex $v_2 \notin (P(x) + x) \cup T_3$ exists adjacent to y_1 and z_2. Hence, (10.54) holds. Define

$$T_4 = T_3 \cup (M(v_2) + v_2).$$

If $v_2 \in P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, by Prop. 10.15, since

$$4k - 14 \geq 18 \geq |T_4|,$$

a vertex $v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x)$ exists. Thus, v_0, v_1, v_2, v_3, v_0 is a closed alternating chain in $G + \{v_2, z_1\}$, as required by Lemma 10.18. Since we have verified the requirements of the lemma, H is a subgraph of G.

Suppose (10.43) holds. This case proceeds just like the proceeding case, except that u' is substituted for v. Thus, we omit the details.

Suppose (10.44) holds. We apply Lemma 10.18 to show that $(z_1\ z_2)(v_0\ v_1\ v_2\ v_3)'\ (u_0\ u_1\ x_2\ x_3\ x_4)'\pi$ embeds H into G, for vertices $v_2, v_3, x_2, x_3, x_4, z_2$ defined below. Let

$$x_3 = x.$$

Thus, we have (10.53), and by (10.44) with u_1 equal to x_1 of Lemma 10.18, we have (10.52). If $v_1 \in M(u_0)$, then let $u_1 = u_0$ in this argument. Let

$$T_1 = \{u_0, v_0, x\} \cup M(u_0, u_1, v_0, v_1, x, z_1).$$

Note that by (10.44),

$$\{u_1, v_1\} = M(z_1) \subseteq T_1,$$

and that since $z_1 \in M(u_1) \cap M(v_1)$ is twice counted,

$$|T_1| \leq 14.$$

By Prop. 10.10, since

$$4k - 8 \geq 14 \geq |T_1|,$$

a vertex $x_4 \in S(x) \cap P(u_0) - T_1$ exists. By Prop. 10.5 and (10.44), there exists a vertex $w_1 \in V(G)$ such that

$$M(u_1) = \{z_1, w_1\}.$$

Let

$$T_2 = T_1 \cup (M(x_4) + x_4).$$

If the vertex of H mapped to z_1 lies in a triangular component of H, then $v_1 = w_1$, and the triangle is embedded onto z_1, u_1, v_1, and (10.41) holds. Hence, without loss of generality, we may assume that w_1, u_1, z_1, v_1 are distinct successive vertices in the image of a path of H. Thus, for the vertices already selected, (10.55) and (10.56) hold.

By Prop. 10.12, since

$$3k - 3 > 17 \geq |T_2|,$$

a vertex $x_2 \in P(x) - T_2$ exists adjacent in G to w_1. Observe that $u_0, u_1, x_2, x_3, x_4, u_0$ is thus a closed alternating chain in $G + \{x_2, z_1\}$. Thus, we have the first of the alternating chains of Lemma 10.18. Define

$$T_3 = T_2 \cup (M(x_2) + x_2).$$

By Prop. 10.12, since

$$3k - 3 > 20 \geq |T_3|,$$

a vertex $z_2 \in P(z_1) - T_3$ exists adjacent in G to x_2. Hence, the second part of (10.54) holds, and (10.57) holds. By Prop. 10.5, and by (10.44), there is a vertex $y_1 \in V(G)$ such that

$$M(v_1) = \{y_1, z_1\}.$$

Since $y_1 \in M(v_1)$, y_1 is adjacent in G to all successors of $v_1 \in S(v_0) - M(x)$. Let

$$T_4 = T_3 \cup M(z_2).$$

By Prop. 10.14, since

$$3k - 7 > 19 \geq |T_4 - \{v_0, x, y_1\}|,$$

a vertex

$$v_2 \notin (P(x) + x) \cup (T_4 - \{v_0, x, y_1\})$$

exists adjacent in G to y_1 and z_2. This verifies (10.54). If $v_2 \in P(v_0) + v_0$, then let $v_3 = v_2$. Otherwise, since

$$4k - 14 \geq 19 \geq |T_4 - (M(v_0) + v_0)|,$$

Prop. 10.15 implies that there is a vertex

$$v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x).$$

Thus, v_0, v_1, v_2, v_3, v_0 is a closed alternating chain in $G + \{v_2, z_1\}$. The other conditions of Lemma 10.18 may be readily verified. Thus, H is a subgraph of G.

This completes the proof of Theorem 10.1.

REFERENCES

1. M. Albertson, B. Bollobás, and S. Tucker, The independence ratio and maximum degree of a graph, to appear in Proc. Seventh South. Conf. on Comb., Graph Theory and Computing (1976).

2. J. Benedict and P. Chinn, Proceedings of the Kalamazoo Graph Theory Conference, to appear.

3. B. Bollobás and S. E. Eldridge, Packings of graphs and applications to computational complexity, to appear in J. Comb. Theory (B).

4. J. A. Bondy, Pancyclic graphs I, J. Comb. Theory 11(1971) 80-84.

5. R. Brooks, On colouring the nodes of a network, Proc. Cambridge Phil. Soc. 37(1941) 194-197.

6. P. A. Catlin, Subgraphs of graphs, I, Discrete Math. 10(1974) 225-233.

7. H. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Hung. 14(1963), 423-439.

8. G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., 2(1952) 69-81.

9. P. Erdős and A. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52(1946) 1087-1091.

10. A. Hajnal and E. Szemeredi, Proof of a conjecture of Erdős, Combinat. Theory and its applications, Balatonfüred (Erdős, Renyi, V. T. Sós, ed.) North Holland, Amsterdam (1970), 601-623.

11. L. Lovász, On decomposition of graphs, Studia Sc. Math. Hungarica 1(1966) 237-288.

12. E. A. Nordhaus and J. W. Gaddum, On complementary graphs, Ann. Math. Monthly, 63(1956) 175-177.

13. Problem session, Proc. 5th Brit. Combinat. Conf., Aberdeen (1976) 690.

14. N. Sauer and J. Spencer, Edge disjoint placement of graphs, to appear in J. Comb. Theory.