EMBEDDING SUBGRAPHS AND COLORING GRAPHS UNDER EXTREMAL DEGREE CONDITIONS

DISSERTATION

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Subgraphs of Graphs, I, Discrete Mathematics 10(1974), 225-233.

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Part I

PRELIMINARIES

1. Notation

All graphs in this thesis are finite and undirected with no loops or multiple edges. Let $V(G)$ denote the set of vertices of G. The edges of G are 2-element subsets of $V(G)$, and the set of all edges of G is $E(G)$. Two vertices u,v are <u>adjacent</u> if $\{u,v\} \in E(G)$.

For any set X, we let $|X|$ denote the cardinality of X. Throughout this thesis, $|V(G)|$ will be denoted by p, and we shall assume that $p \geq 1$.

The number of edges incident with a vertex $v \in V(G)$ is called the <u>degree</u> of v in G, and is denoted $\deg_G(v)$. We define

$$\Delta(G) = \max_{v \in V(G)} \deg_G(v)$$

and

$$\delta(G) = \min_{v \in V(G)} \deg_G(v).$$

The <u>complement</u> of G, denoted G^c, is the graph on the same vertex set $V(G)$, in which $\{u,v\} \in E(G^c)$ if and only if $\{u,v\} \notin E(G)$, where $u,v \in V(G)$. Clearly, for any graph G,

$$\Delta(G^c) + \delta(G) + 1 = p.$$

For two graphs G and H with $|V(H)| \leq |V(G)|$, an embedding of H into G is an injection .

$$\pi : V(H) \longrightarrow V(G)$$

that maps edges of H into edges of G. If such an embedding exists, we say that H is a subgraph of G. Note that when $|V(H)| = |V(G)|$, H is a subgraph of G if and only if G^c is a subgraph of H^c.

Brackets will be used with two meanings, depending upon their context. For any rational number r, $[r]$ denotes the greatest integer less than or equal to r. For a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X: thus, $V(G[X]) = X$ and if $u, v \in X$, then $\{u,v\} \in E(G[X])$ if and only if $\{u,v\} \in E(G)$. We denote by $G - X$ the graph $G[V(G) - X]$.

A complete graph on n vertices is a graph on n vertices in which any pair of distinct vertices are adjacent. Such a graph will be denoted by K_n. A complete bipartite graph on disjoint sets of n and m vertices is the graph on these vertices in which each vertex in the n-set is adjacent to every vertex in the m-set. Such a graph is denoted $K_{n,m}$.

A maximal complete subgraph induced by some vertices of a graph is called a clique. A maximal complete bipartite induced subgraph is called a biclique.

A set X of vertices is stable if G[X] is edgeless. The maximum cardinality of all stable sets $X \subseteq V(G)$ is denoted $\beta(G)$, and is called the stability number of G. The maximum number of vertices in a clique of G, denoted $\theta(G)$, is called the clique number of G. Clearly,

$$\theta(G) = \beta(G^c); \qquad \theta(G^c) = \beta(G).$$

A coloring of G is a partition of V(G) into stable subsets, where the partition is unordered and admits null sets. A set $X \subseteq V(G)$ is monochromatic in a coloring of G if all vertices of X have the same color: i.e., they lie in the same set in the coloring partition. The chromatic number $\chi(G)$ of G is the fewest possible number of sets in a coloring of G.

A path in G is a sequence of vertices $v_0, v_1, \ldots, v_n$ in V(G) for $n \geq 1$ such that

(1.1) $v_i = v_j$ implies either $i = j$ or $\{i, j\} = \{0, n\}$;

(1.2) for $i = 1, 2, \ldots, n$, v_i is adjacent in G to v_{i-1}.

The vertices v_0 and v_n are said to be joined by the path. If $v_0 = v_n$, we say that the path is closed; otherwise, the path is open. A graph is connected if any two vertices are joined by a path. A component of G is a maximal connected subgraph of G. A vertex of a connected graph is a cutvertex if its removal disconnects the graph. A polygon is a subgraph determined by a set of vertices and edges joining consecutive vertices in a closed path.

The girth is the number of edges of the polygon. A polygon with odd girth is an odd polygon. An arc is a subgraph determined by the set of vertices and edges joining consecutive vertices in an open path. An odd arc is an arc with an odd number of edges.

A tree is a connected graph having no polygons.

A θ-graph is a graph consisting of three distinct arcs, joining the same two vertices and having no other common vertices.

To simplify notation, we shall denote the singleton set $\{x\}$ by x.

Given a set X and a subset $\{x_1,\ldots,x_n\}$, let $(x_1\ x_2\ \ldots\ x_n)$ denote the cyclic permutation that sends x_i to x_{i+1}, $1 \leq i \leq n$, that sends x_n to x_1, and that fixes all other elements of X. Given a permutation α: $X \longrightarrow X$ and a function π: $Y \longrightarrow X$, for sets X and Y, we denote by $\alpha\pi$ the composition of α and π which maps $y \in Y$ to $\alpha(\pi(y)) \in X$.

Given a set X and a finite sequence $x_1,x_2,\ldots,x_n$ of members of X, such that $x_i = x_j$, $i < j$ imply $x_i = x_{i+1} = \ldots = x_j$, let $(x_1\ x_2\ \ldots\ x_n)'$ denote the cyclic permutation obtained by deleting from $x_1,x_2,\ldots,x_n$ the terms which have previously appeared in the sequence.

2. Introduction

Two problems are considered in this dissertation. They concern somewhat separate topics, but both depend upon degree constraints, and there are several points of overlap. First, we consider the problem of estimating the chromatic number $\chi(G)$, knowing $\Delta(G)$ and $\theta(G)$. Then, we consider the problem of giving sufficient conditions, in terms of $\Delta(H)$ and $\Delta(G^c)$, for a graph H on p vertices to be a subgraph of a graph G, also on p vertices.

The basic result in the literature on the coloring problem is Brooks' Theorem [5]:

Theorem 2.1 Let G be a graph with maximum degree $\Delta(G)$. We have

(2.1) $\chi(G) \leq \Delta(G) + 1.$

If $\Delta(G) = 2$, then equality holds in (2.1) if and only if G contains an odd polygon. If $\Delta(G) \neq 2$, then equality holds if and only if G contains a clique $K_{\Delta(G)+1}$.

Note that if $\Delta(G) = 2$, an odd polygon of G is necessarily a connected component of G. Also, a clique $K_{\Delta(G)+1}$ is necessarily a component of G. Such components, which force equality in (2.1), are called $B_{\Delta(G)}$-components.

Since each component of a graph can be colored independently, we can assume without loss of generality, that G is connected.

We give a proof of Brooks' Theorem by induction on $\Delta(G)$, and in so doing, we obtain new information. For instance, we show that if G is not a $B_{\Delta(G)}$-component, then there is a coloring of G in $\Delta(G)$ colors in which some monochromatic set contains $\beta(G)$ vertices. Also, we characterize those connected graphs G for which there is a coloring of G in $\Delta(G)$ colors such that some monochromatic set consists solely of vertices of degree $\Delta(G)$.

In section 4 we consider the problem of partitioning the vertices of a graph into sets $X_1, X_2, \ldots, X_n$ such that the numbers $\Delta(G[X_i])$, $i = 1, 2, \ldots, n$ satisfy various constraints. One result will be used for a problem on subgraphs. Another result is a new proof of a partition theorem of Lovász [11].

We combine, in section 5, this partition theorem of Lovász with Brooks' Theorem to give an estimate of $\chi(G)$ in terms of $\Delta(G)$ and $\theta(G)$. The result improves (2.1) when $\theta(G) < \frac{1}{2}\Delta(G)$.

In section 6 we consider further the interrelationship between $\chi(G)$, $\Delta(G)$ and $\theta(G)$.

In [6], we considered the problem of giving a sufficient condition, based upon $\Delta(H)$ and $\Delta(G^c)$, for

H to be a subgraph of G. We continue here to obtain sharper results.

Our first result, which has recently been independently obtained by Sauer and Spencer [14], is that if G and H are graphs on p vertices satisfying

$$2\Delta(G^c)\Delta(H) \leq p-1,$$

then H is a subgraph of G. This is best possible only when $\Delta(G^c) = 1$ or $\Delta(H) = 1$. We continue, in section 7, by discussing a conjectured improvement of this result that would be best possible if true, and we consider various special cases treated in the literature.

In section 8, we give a slightly sharper result when $\Delta(H) = 2$ whose proof is not long.

In section 10, we show that if $\Delta(H) = 2$ and if

$$\Delta(G^c) \leq \frac{1}{3}p - \max(9, \frac{3}{2}p^{1/3}),$$

then H is a subgraph of G. The coefficient $\frac{1}{3}$ is best possible. However, the proof is quite long. In the special case where every component of H is either K_3, K_2, or K_1, we obtain an even sharper result in section 9. We show that if $\Delta(G^c) \leq \frac{p-1}{3}$ and if such a graph H is not a subgraph of G, then G lies in one of two classes which do not have H as a subgraph. We characterize these classes.

Part II

CHROMATIC NUMBER

3. Brooks' graph-coloring theorem and the stability number

In this section, we shall consider a connected graph H, with at least one edge. To simplify notation, we denote $\Delta(H)$ by h.

A maximum stable subset of the set of vertices of degree h will be called a superstable set.

A B_h-component of H was defined in section 2. The equivalence of (3.4) and (3.6) of Theorem 3.2 below is Brooks' Theorem (Theorem 2.1).

Albertson, Bollobás, and Tucker [1] showed first that with two exceptions H_1 and H_2, defined below, every graph H with $\Delta(H) = h$ and with no subgraph K_h has stability number

$$\beta(H) > |V(H)|/h,$$

and they conjectured that such graphs H have an h-coloring in which some monochromatic set has more than $|V(H)|/h$ vertices. Second, they proved this conjecture for graphs that are not regular of degree h. Theorem 3.2, combined with the first result of Albertson, Bollobás, and Tucker shows that this conjecture is true, even for regular graphs.

The two exceptional graphs, H_1 and H_2, may be defined as follows: let $V(H_1)$ be the integers modulo 8, and let $\{v,w\} \in E(H_1)$ if and only if

$$v - w \equiv 1, 2, 6, \text{ or } 7 \pmod 8.$$

Let $V(H_2)$ be the integers modulo 10, and let $\{v,w\} \in E(H_2)$ if and only if

$$v - w \equiv 1, 4, 5, 6, \text{ or } 9 \pmod{10}.$$

A Brooks tree is any graph H with $\Delta(H) = h$ that arises from a tree T satisfying $\Delta(T) \leq h$ by the replacement of each vertex of T with

(a) an odd polygon if $h = 3$;

(b) a clique K_h if $h \neq 3$,

such that if x and y are adjacent vertices of T, then the polygons or cliques substituted for x and y are joined by an edge whose removal disconnects H. Thus, K_2 is the only Brooks tree with $h = 1$; odd arcs with at least 3 edges are the only Brooks trees with $h = 2$; and if $h \geq 3$, then a Brooks tree is not a tree in the usual sense of the word.

Theorem 3.1 Let H be a connected graph with $\Delta(H) = h \geq 1$. The following are equivalent:

(3.1) H is a B_h-component, or a Brooks tree;

(3.2) There is no superstable set S such that $H - S$ can be colored in $h - 1$ colors;

(3.3) There is no stable set S of vertices of degree h such that $H - S$ can be colored in $h - 1$ colors.

We also have

<u>Theorem 3.2</u> Let H be a connected graph with $\Delta(H) = h \geq 1$. The following are equivalent:

(3.4) H is a B_h-component;

(3.5) There is no maximum stable set S, such that $H - S$ can be colored in $h - 1$ colors;

(3.6) There is no h-coloring of H.

<u>Proof of Theorem 3.2 from Theorem 3.1</u>: For $\Delta(H) \leq 2$, the theorem is easily verified. Assume therefore, that $\Delta(H) \geq 3$.

We show that if (3.1), (3.2), and (3.3) are equivalent for $\Delta(H) = h$, then (3.4), (3.5), and (3.6) are also equivalent for $\Delta(H) = h$. Since (3.4) implies (3.6) and (3.6) implies (3.5), it suffices to prove that (3.5) implies (3.4) if (3.1), (3.2), and (3.3) are equivalent.

Adjoin to H a set V of $\Sigma(h - \deg_H(v))$ vertices disjoint from V(H), where the sum runs over all $v \in V(H)$. We join each vertex v of H to exactly $h - \deg_H(v)$ vertices of V, such that no vertex of V is joined to more than one vertex of H. Denote the resulting graph H'. Then, (3.7) $H'[V(H)] = H$;

(3.8) Any $v \in V(H)$ has degree h in H';

(3.9) Any $v \in V$ has degree 1 in H'.

By (3.7) and (3.8), a superstable set S in H' is a maximum stable set in H. Hence, (3.5) for H implies (3.2) for H', whence by (3.1), either H' is a B_h-component, or it is a Brooks tree. Since Brooks trees have vertices of degree $h-1$, conditions (3.8), (3.9), and $h \geq 3$ imply that H' is not a Brooks tree. Thus, H' is a B_h-component, and therefore, has no vertices of degree 1, whence $H = H'$. This proves (3.4), and thus the equivalence of (3.4), (3.5), and (3.6). Hence, Theorem 3.2 follows from Theorem 3.1.

Proof of Theorem 3.1: Again, we may suppose that $h \geq 3$. Since (3.1) implies (3.3) and (3.3) implies (3.2), it suffices to show that (3.2) implies (3.1).

Suppose inductively that the theorem is true for all graphs G with $\Delta(G) < h$. Then Theorem 3.2 is true for such graphs G. Let H be a graph with $\Delta(H) = h$ such that H does not satisfy (3.1), and such that for any superstable set S, $H - S$ has no $(h-1)$-coloring. For a given superstable set S, Theorem 3.2 and

$$\Delta(H - S) \leq h - 1$$

imply that either $H - S$ can be colored in $h - 1$ colors, or $H - S$ has a B_{h-1}-component. We have already precluded the first possibility. Hence, $H - S$ has a B_{h-1}-component. Without loss of generality, we shall choose S to be a superstable set that minimizes the number of B_{h-1}-components in $H - S$.

Suppose that a vertex $s \in V(H)$ is in no B_{h-1}-component in $H - S$, regardless of the choice of a superstable set S that minimizes the number of B_{h-1}-components in $H - S$. Since H is connected, such a vertex s exists that is adjacent to a vertex v lying in a B_{h-1}-component C of $H - S$, for some such S. Since the only vertex not in C that is adjacent to v lies in S, we must have $s \in S$. Then $S + v - s$ is a superstable set, and either $H - (S + v - s)$ has one fewer B_{h-1}-component than $H - S$, contrary to the choice of S, or s lies in a B_{h-1}-component of $H - (S + v - s)$, contrary to the choice of s. Hence, by contradiction, all vertices of H lie in B_{h-1}-components of $H - S$, for suitable S.

Let P be a polygon in H with the property that there is no superstable set S such that a B_{h-1}-component of $H - S$ contains P. If $h = 3$, any polygon of even girth will do; otherwise, any polygon not contained in a clique suffices. We will show that if H is not a B_h-component or a Brooks tree, then such a P must exist.

If P does not exist, then

(3.10) If P' is a polygon in H and if $h = 3$, then P' has odd girth

and

(3.11) If P' is a polygon in H and $h \geq 4$, then P' is contained in a clique.

Suppose, by way of contradiction, that there are distinct overlapping subgraphs C_1 and C_2 of H, where C_i is a B_{h-1}-component of $H - S_i$, for some superstable set S_i. If $h \geq 4$, then C_1 and C_2 are cliques on h vertices each. Since C_1 and C_2 overlap, $\Delta(H) = h$ forces

$$|V(C_1) \cup V(C_2)| \leq h + 1.$$

Since C_1 and C_2 are distinct, we have equality, and hence $H[V(C_1) \cup V(C_2)]$ is either isomorphic to K_{h+1} or to K_{h+1} minus an edge. In the first case, H is a B_h-component. In the second case, let P' be a polygon on 4 vertices in $H[V(C_1) \cup V(C_2)]$ containing the 2 non-adjacent vertices. This violates (3.11). If $h = 3$, then C_1 and C_2 are overlapping odd polygons, and $h < 4$

forces them to overlap in an edge. Then $C_1 \vee C_2$ contains a θ-graph, and hence an even polygon. Thus, (3.10) is violated. Hence, if P does not exist, then, since each vertex of H lies in a B_{h-1}-component of $H - S$ for a suitable superstable set S, V(H) can be partitioned into sets $V_1, V_2, \ldots, V_n$, such that $H[V_i]$ is a B_{h-1}-component of $H - S$, for suitable superstable S. All polygons of G are contained in these $H[V_i]$. Moreover, H is connected, and so it is easily seen in this case that if (3.10) and (3.11) hold, then H must be a Brooks tree or a B_h-component. This is contrary to assumption, and we may therefore conclude that P does exist.
To prove the theorem, we will derive a contradiction from the existence of P.

Let C_0 be a B_{h-1}-component of $H - S_0$, such that C_0 intersects P, and such that S_0 is superstable and chosen to minimize the number of B_{h-1}-components in $H - S_0$. Since the degree of any vertex of C_0 in $H - S_0$ is $h - 1$, and since $\Delta(H) = h$, an edge of P lies in $E(C_0)$. Since P is not contained in C_0, which is an induced subgraph of H, an edge of P lies outside $E(C_0)$. Therefore, there is a vertex v of $V(P) \cap V(C_0)$ having one incident edge in $E(C_0)$ and the other incident edge

$\{v, s\}$ outside $E(C_0)$. Since C_0 is a component of $H - S_0$, we have $s \in S_0$.

Define a sequence $v_1, s_1, v_2, s_2, \ldots, v_m, s_m$ of vertices along P as follows: Let

$$v_1 = v; \qquad s_1 = s;$$

$$S_1 = S_0 + v_1 - s_1.$$

For each $i = 1, 2, \ldots, m-1$, there is a superstable set

$$S_i = S_{i-1} + v_i - s_i$$

and a (unique) B_{h-1}-component C_i of $H - S_i$ containing s_i. If for some i, s_i is not in a B_{h-1}-component of $H - S_i$, then $H - S_i$ has fewer B_{h-1}-components than $H - S_0$, contrary to our choice of S_0. The polygon P intersects C_i in a path starting at s_i and ending at a vertex of S_i, which we shall call v_{i+1}. Thus, we have determined a vertex $s_{i+1} \in V(P) \cap S_i$ that is adjacent in P to v_{i+1} and is not in C_i. Since v_{i+1} is adjacent to $h - 1$ vertices in C_i also, $\deg_H(v_{i+1}) = h$. Thus, since S_i is superstable,

$$S_{i+1} = S_i + v_{i+1} - s_{i+1}$$

is also a superstable set. We terminate the sequence at the first vertex s_m $(m \geq 1)$ that is adjacent to a vertex of the original B_{h-1}-component C_0 of $H - S_0$. To see that s_m exists, note that P determines a closed path, and the first vertex along that path after v and s that is adjacent to a vertex of the original B_{h-1}-component is necessarily in S_0, and hence in S_i for each $i < m$.

Of course, since $s_m \in S_{m-1}$ is the first vertex in the sequence to be adjacent to a vertex of $V(C_0)$, the vertices of $V(C_0 - v)$ have not been moved into the superstable set S_i, as i runs from 0 to $m - 1$, and no vertices adjacent to vertices of C_0 have been moved out of the superstable set. Thus, in the B_{h-1}-component of $H - S_m$ containing s_m and $C_0 - v$, any vertices other than s_m or $V(C_0 - v)$ would be adjacent to s_m only. But no vertex of a B_{h-1}-component is a cutvertex, and so s_m and $V(C_0 - v)$ together induce a B_{h-1}-component of $H - S_m$. Therefore, we must have

$$N(s_m) - v_m = N(v) - s,$$

where N(v) denotes the set of vertices of H adjacent to v.

If C_0 is a polygon of girth at least 5, then s_m is adjacent to two nonadjacent vertices x_1, x_2 of degree $h = 3$ that comprise $N(v) - s$. Since s_m is the only vertex in S_0 to which x_1 and x_2 are adjacent, $S_0 \cup \{x_1, x_2\} - s_m$ is a bigger superstable set than S_0, contrary to the maximality of S_0.

If C_0 is a clique K_h, then s_m is adjacent to every vertex of $C_0 - v_1$. If v_1 and s_m are adjacent, then $m = 1$, and $V(C_0) + s_m$ induces a clique K_{h+1} in H. Since H is connected, K_{h+1} is necessarily all of H, a case excluded since (3.1) is false. Suppose, therefore, that s_m and v_1 are not adjacent. Let x be a member of the equal sets $V(C_m - s_m) = V(C_0 - v)$. Then $H - (S_0 + x - s_m)$ has fewer B_{h-1}-components than $H - S_0$, and $S_0 + x - s_m$ is a superstable set. Since this contradicts the choice of H, P does not exist. But, as we have seen, this contradicts the assumption that H is a B_h-component or a Brooks tree. This proves the theorem.

4. Some partition theorems

We consider the problem of partitioning the vertex set of a graph so that the subgraphs induced by the subsets of vertices will satisfy various constraints on the degree of their vertices.

Given sets $X, Y \subseteq V(G)$, we denote by $E(X,Y)$ the set of edges in $E(G)$ with one end in X and the other end in Y. Let $E^c(X,Y)$ denote the set of edges in $E(G^c)$ with one end in X and the other end in Y.

Given a partition $X_1 \cup X_2$ of $V(G)$, we simplify notation by writing G_i for the subgraph $G[X_i]$ induced by X_i, where $i = 1,2$.

Lovász [11] proved a variation on the first theorem below, except that he maximized an expression different than $f_1(X_1, X_2)$.

Let h_1 and h_2 be integers, and let

$$f_1(X_1, X_2) = |E(X_1, X_2)| + h_1|X_1| + h_2|X_2|.$$

Theorem 4.1 Let G be a graph with maximum degree $\Delta(G) \geq 1$, and let h_1, h_2 be nonnegative integers such that

$$\Delta(G) = h_1 + h_2 + 1.$$

If $X_1 \cup X_2$ is a partition of $V(G)$ that maximizes f_1, then for $i = 1,2$, X_i is nonempty, and

$$\Delta(G_i) \leq h_i.$$

Proof: Of X_1, X_2, at least one set, say X_1, is nonempty. Later, we show that X_2 is also nonempty, whence the following argument applies also to X_2. Let $x \in X_1$. By hypothesis,

$$\begin{aligned} 0 &\leq f_1(X_1,X_2) - f_1(X_1 - x, X_2 + x) \\ &= |E(X_1,X_2)| + h_1|X_1| + h_2|X_2| - |E(X_1 - x, X_2 + x)| \\ &\qquad - h_1(|X_1| - 1) - h_2(|X_2| + 1) \\ &= |E(x,X_2)| - |E(x,X_1)| + h_1 - h_2. \end{aligned}$$

We add $2\deg_{G_1}(x) = 2|E(x,X_1)|$ to each side and get

$$\begin{aligned} 2\deg_{G_1}(x) &\leq |E(x,X_2)| + |E(x,X_1)| + h_1 - h_2 \\ &= \deg_G(x) + h_1 - h_2 \\ &\leq (h_1 + h_2 + 1) + h_1 - h_2 \\ &= 2h_1 + 1. \end{aligned}$$

Dividing by 2 and observing that the left side is an integer, we get

$$\deg_{G_1}(x) \leq h_1.$$

Since $x \in X_1$ is arbitrary, we have

$$\Delta(G_1) \leq h_1 < \Delta(G),$$

whence, X_1 is not $V(G)$. Thus, X_2 is also not empty, and the theorem follows.

Corollary 4.2 (Lovász [11]) Let G be a graph with $\Delta(G) = h$, and let $h_1, h_2, \ldots, h_n$ be nonnegative integers satisfying

$$h = h_1 + h_2 + \ldots + h_n + n - 1.$$

Then there is a partition $V(G) = X_1 \cup X_2 \cup \ldots \cup X_n$ such that for $i \leq n$, if X_i is not empty, then

$$\Delta(G[X_i]) \leq h_i.$$

Proof: Let Theorem 4.1, where $n = 2$, be a basis for induction. Assume inductively that this corollary is true for $n - 1$, and write

$$h = h_1 + (h_2 + \ldots + h_n + (n-1) - 1) + 1.$$

Theorem 4.1 asserts that there is a partition $X_1 \cup (V(G) - X_1)$ such that

$$\Delta(G[X_1]) \leq h_1$$

$$\Delta(G - X_1) \leq h_2 + \ldots + h_n + (n-1) - 1.$$

By the induction hypothesis, there is a partition $X_2 \cup \ldots \cup X_n$ of $V(G) - X_1$ such that

$$\Delta(G[X_i]) \leq h_i,$$

for $i = 1, 2, \ldots, n$. This proves the corollary.

<u>Conjecture</u>: Let G be a graph on p vertices. If neither G nor G^c is edgeless, then there are partitions $X_1 \cup X_2$ and $Y_1 \cup Y_2$ of V(G) such that

$$\Delta(G[X_1]) + \Delta(G[X_2]) + \Delta(G^c[Y_1]) + \Delta(G^c[Y_2]) \leq p - 3.$$

If G is regular, then this conjecture follows easily from Theorem 4.1.

Suppose that the conjecture is true. It is easily verified that for any graph G,

$$\chi(G) \leq \Delta(G) + 1.$$

Thus, the inequality of the conjecture implies

$$\chi(G[X_1]) + \chi(G[X_2]) + \chi(G^c[Y_1]) + \chi(G^c[Y_2]) \leq p + 1.$$

Therefore, for any graph G,

$$\chi(G) + \chi(G^c) \leq p + 1.$$

Since this inequality is the theorem of Nordhaus and Gaddum [12], the conjecture, if true, would generalize their theorem.

A nontrivial partition $X_1 \cup X_2$ of V(G) is a partition in which both X_1 and X_2 are nonempty.

For any partition $X_1 \cup X_2$ of $V(G)$ we write

$$G_i = G[X_i], \qquad i = 1,2,$$

and

$$p_i = |X_i|, \qquad i = 1,2,$$

and define, for $c \in (0,1]$,

$$f_2(X_1,X_2) = |E(X_1,X_2)| + \tfrac{1}{2}cp_1^2 + \tfrac{1}{2}cp_2^2.$$

Theorem 4.3 Let G be a graph with

$$\Delta(G) = c(p-1)$$

for $c \in (0,1]$ and $p \geq 2$. For any partition $X_1 \cup X_2$ of $V(G)$ such that

(4.1) f_2 is maximized, and

(4.2) $\tfrac{1}{2}c(p_1^2 + p_2^2)$ is minimized, subject to (4.1),

it follows that

(4.3) $X_1 \cup X_2$ is a nontrivial partition;

and for $i = 1,2$,

(4.4) $\Delta(G_i) \leq c(p_i - 1)$.

Proof: Define the linear function

(4.5) $c(t) = c - t$,

where $t \geq 0$. Thus,

(4.6) $\Delta(G) = c(p-1) = c(t)(p-1) + t(p-1)$.

For any partition $X_1 \cup X_2$ of $V(G)$ and any $t \geq 0$, define

$$F_t(X_1,X_2) = |E(X_1,X_2)| + \tfrac{1}{2}c(t)(p_1^2 + p_1^2).$$

Thus, for X_1 and X_2 fixed, F_t is a linear function of t with F-intercept $f_2(X_1,X_2)$ and with slope $-\frac{1}{2}(p_1^2 + p_2^2)$. Moreover, F_0 is equal to f_2.

Therefore, if $X_1 \cup X_2$ satisfies (4.1), then for any other partition $Y_1 \cup Y_2$ of V(G),

$$F_0(X_1,X_2) \geq F_0(Y_1,Y_2).$$

Also, (4.2) assures that if $Y_1 \cup Y_2$ is another partition that maximizes $f_2(X_1,X_2)$, then

$$F_t(X_1,X_2) \geq F_t(Y_1,Y_2).$$

Thus, the only way that we could have

$$F_t(X_1,X_2) < F_t(Y_1,Y_2)$$

if (4.1) and (4.2) hold is if

$$F_0(X_1,X_2) > F_0(Y_1,Y_2)$$

and if the slope of $F_t(X_1,X_2)$ is strictly less than that of $F_t(Y_1,Y_2)$, and t is sufficiently large. Thus, for $t \geq 0$ sufficiently close to 0, if (4.1) and (4.2) hold, then $X_1 \cup X_2$ also maximizes F_t. We shall consider t to be small enough so that $X_1 \cup X_2$ also maximizes F_t.

Reversing the indices if necessary, we may suppose without loss of generality that X_1 is nonempty. Let $x \in X_1$. We have

$$(4.8) \qquad \begin{aligned} 0 &\le F_t(X_1,X_2) - F_t(X_1 - x, X_2 + x) \\ &= |E(X_1,X_2)| + \tfrac{1}{2}c(t)(p_1^2 + p_2^2) \\ &\quad - |E(X_1 - x, X_2 + x)| \\ &\quad - \tfrac{1}{2}c(t)((p_1 - 1)^2 + (p_2 + 1)^2) \\ &= |E(x,X_2)| - |E(x,X_1)| + c(t)p_1 \\ &\quad - c(t)p_2 - c(t). \end{aligned}$$

We add $2\deg_{G_1}(x) = 2|E(x,X_1)|$ to each side and get

$$\begin{aligned} 2\deg_{G_1}(x) &\le \deg_G(x) + c(t)p_1 - c(t)p_2 - c(t) \\ &\le (c(t) + t)(p_1 + p_2 - 1) + c(t)p_1 \\ &\qquad - c(t)p_2 - c(t) \\ &= 2c(t)(p_1 - 1) + t(p - 1). \end{aligned}$$

We divide by 2 and substitute for c(t) to get

$$(4.9) \qquad \begin{aligned} \deg_{G_1}(x) &\le c(t)(p_1 - 1) + \tfrac{1}{2}t(p - 1) \\ &= c(p_1 - 1) - t(p_1 - 1) + \tfrac{1}{2}t(p - 1) \\ &= c(p_1 - 1) + \tfrac{1}{2}t(p - 2p_1 + 1). \end{aligned}$$

If $G_1 = G$, then $p_1 = p$, whence by (4.9), if x is a vertex of maximum degree in G, then

$$\begin{aligned} \deg_G(x) &= \deg_{G_1}(x) \\ &\le c(p - 1) + \tfrac{1}{2}t(1 - p) \\ &< c(p - 1) \\ &= \deg_G(x), \end{aligned}$$

a contradiction. Hence, (4.3) holds, and (4.9) applies to either set X_1 or X_2. Since (4.9) holds for $t = 0$, (4.4) follows.

Let $X_1 \cup X_2$ be a nontrivial partition that maximizes $f_j(X_1,X_2)$, with $j=1$ in Theorem 4.1 or with $j=2$ in Theorem 4.3. If Theorem 4.3 applies, assume also that (4.2) holds. If $x_1 \in X_1$ and $x_2 \in X_2$ have the property that

(4.10) $$|E(X_1,X_2)| = |E(X_1+x_2-x_1, X_2+x_1-x_2)|,$$

then $(X_1+x_2-x_1) \cup (X_2+x_1-x_2)$ is also a partition of $V(G)$ such that the above conditions hold. Any pair x_1,x_2 of vertices satisfying condition (4.10) are called <u>interchangeable</u>. If $x_1 \in X_1$ and $x_2 \in X_2$ are interchangeable vertices, then $G[X_1+x_2-x_1]$ and $G[X_2+x_1-x_2]$ satisfy the same conclusions in Theorems 4.1 and 4.3 that apply to $G[X_1]$ and $G[X_2]$.

<u>Theorem 4.4</u> If in Theorem 4.1 or 4.3 $x_1 \in X_1$ and $x_2 \in X_2$ are two adjacent vertices such that

(4.11) $$\deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \Delta(G) - 1,$$

then x_1 and x_2 are interchangeable, and we have

$$\deg_{G_i}(x_i) = \begin{cases} h_i & \text{in Theorem 4.1;} \\ [c(p_i-1)] & \text{in Theorem 4.3,} \end{cases}$$

and

$$\deg_G(x_i) = \Delta(G).$$

If x_3 is another vertex that is interchangeable with x_1, then x_2 and x_3 are adjacent in G.

<u>Proof</u>: Let $x_1 \in X_1$ and $x_2 \in X_2$ be adjacent vertices satisfying (4.11), where $X_1 \cup X_2$ is a partition of $V(G)$ that maximizes $f_1(X_1, X_2)$ in Theorem 4.1 or maximizes $f_2(X_1, X_2)$ and satisfies (4.2) in Theorem 4.3. We have

$$
\begin{aligned}
(4.12) \quad |E(X_1 + x_2 - x_1, X_2 + x_1 - x_2)| &= |E(X_1, X_2)| \\
&\quad + \deg_{G_1}(x_1) + \deg_{G_2}(x_2) \\
&\quad - |E(x_1, X_2 - x_2)| - |E(x_2, X_1 - x_1)| \\
&= |E(X_1, X_2)| + 2\deg_{G_1}(x_1) + 2\deg_{G_2}(x_2) \\
&\quad - |E(x_1, V(G) - x_2)| - |E(x_2, V(G) - x_1)| \\
&= |E(X_1, X_2)| + 2(\Delta(G) - 1) - (\deg_G(x_1) - 1) \\
&\quad - (\deg_G(x_2) - 1) \qquad \text{(by (4.11))} \\
&\geq |E(X_1, X_2)| .
\end{aligned}
$$

By the maximality of $f_j(X_1, X_2)$ in Theorems 4.1 and 4.3, $|E(X_1, X_2)|$ cannot be less than $|E(X_1 + x_2 - x_1, X_2 + x_1 - x_2)|$. Hence, (4.12) holds with equality. Thus, x_1 and x_2 are interchangeable. Also, since (4.12) holds with equality,

$$\Delta(G) - 1 = \deg_G(x_i) - 1 \qquad (i = 1, 2),$$

whence,

$$\deg_G(x_i) = \Delta(G).$$

Observe that if (4.11) holds, then $\deg_{G_1}(x_1)$ and $\deg_{G_2}(x_2)$ attain the upper bound specified by Theorem 4.1 or 4.3, whichever is applicable. For instance, from (4.11) and from (4.4) of Theorem 4.3,

$$\begin{aligned}\Delta(G) - 1 &= \deg_{G_1}(x_1) + \deg_{G_2}(x_2) \\ &\le \Delta(G_1) + \Delta(G_2) \\ &\le c(p_1 - 1) + c(p_2 - 1) \\ &= c(p - 1) - c \\ &= \Delta(G) - c \\ &< \Delta(G).\end{aligned}$$

Thus, since $\Delta(G)$ is an integer,

$$(4.13) \quad \deg_{G_i}(x_i) = \Delta(G_i) = [c(p_i - 1)],$$

for $i = 1$ and 2. In Theorem 4.1, we can more easily obtain

$$(4.14) \quad \deg_{G_i}(x_i) = h_i \qquad (i = 1, 2).$$

If, contrary to the conclusion of Theorem 4.4, x_2 is not adjacent to x_3, then in $G[X_2 + x_1 - x_3]$, x_2 is adjacent to x_1 and to h_2 or $[c(p_2 - 1)]$, respectively, other vertices in $G[X_2 + x_1 - x_3]$, depending upon whether we consider Theorem 4.1 or Theorem 4.3, respectively. However, we have

$$\Delta(G[X_2 + x_1 - x_3]) \le \begin{cases} h_2 & \text{in Theorem 4.1;} \\ [c(p_2 - 1)] & \text{in Theorem 4.3,} \end{cases}$$

since x_1 and x_2 are interchangeable, and so we have a contradiction. Thus, x_2 must be adjacent to x_3.

We shall use the following result in section 9.

Theorem 4.5 Let G be a graph with $p \geq 2$ and

(4.15) $\delta(G) = c(p-1)$

for some $c \in [0,1)$. There is a nontrivial partition $X_1 \cup X_2$ of $V(G)$ which maximizes

(4.16) $f_3(X_1,X_2) = \frac{1}{2}(1-c)(p_1^2+p_2^2) - |E(G_1^c)| - |E(G_2^c)|$

and satisfies

(4.17) $\delta(G_i) \geq c(p_i - 1)$,

for $i = 1$ and 2. Furthermore, suppose $x_1 \in X_1$ and $x_2 \in X_2$ are adjacent in G^c and satisfy

(4.18) $\deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \delta(G)$.

Then x_1 and x_2 are interchangeable,

(4.19) $\deg_G(x_1) = \deg_G(x_2) = c(p-1)$,

and the set of vertices in X_{3-i} interchangeable with x_i are adjacent in G_{3-i}^c to x_{3-i}.

Proof: By (4.15), G^c satisfies

(4.20) $\Delta(G^c) = (1-c)(p-1)$

for some $c \in [0,1)$. Note that a partition that maximizes $f_3(X_1,X_2)$ also maximizes

$$f_3(X_1,X_2) + |E(G^c)| = |E^c(X_1,X_2)| + \frac{1}{2}(1-c)(p_1^2+p_2^2),$$

which is $f_2(X_1,X_2)$ with $1-c$ in place of c and E^c in place of E. Hence, by Theorem 4.3, there is a nontrivial partition of $X_1 \cup X_2$ of $V(G)$ that maximizes $f_3(X_1,X_2)$

such that

$$(4.21) \qquad \Delta(G_i^c) \leq (1-c)(p_i - 1),$$

by (4.4), whence (4.17) follows.

If $x_1 \in X_1$ and $x_2 \in X_2$ satisfy (4.18), then

$$\deg_{G_1^c}(x_1) + \deg_{G_2^c}(x_2) = \Delta(G^c) - 1,$$

whence (4.11) of Theorem 4.4 holds for G^c. The remaining conclusions of Theorem 4.5 follow directly from Theorem 4.4 applied to G^c.

5. A bound on the chromatic number of a graph.

In this section we combine Theorem 2.1 of Brooks [5] and Corollary 4.2 of Lovász [11] to give an upper bound on the chromatic number of a graph G, in terms of $\Delta(G)$ and $\theta(G)$.

<u>Theorem 5.1</u> If G is a graph with no complete subgraphs on r vertices, where $r \geq 4$, then

$$\chi(G) \leq \Delta(G) + 1 - [(\Delta(G) + 1)/r].$$

<u>Proof</u>: To simplify notation, let

$$n = [(\Delta(G) + 1)/r].$$

If $n = 0$, then Theorem 5.1 follows. Thus, we can assume that $n > 0$.

By Corollary 4.2, there is a partition of V(G) into n sets $X_1, X_2, \ldots, X_n$, such that if X_i is nonempty, then

$$\Delta(G[X_i]) \leq r - 1 \qquad \text{for } i = 1, 2, \ldots, n-1,$$

and such that if X_n is nonempty then

$$\Delta(G[X_n]) \leq \Delta(G) - r(n-1).$$

Since G contains no complete subgraphs on r vertices, neither do the subgraphs $G[X_i]$, for all $i \leq n$. Hence, by these inequalities and Brooks' Theorem,

$$\chi(G[X_i]) \leq r-1 \qquad \text{for } i=1,2,\ldots,n-1,$$

and

$$\chi(G[X_n]) \leq \Delta(G) - r(n-1).$$

The latter inequality follows because by definition of n,

$$\Delta(G) - r(n-1) \geq r-1,$$

whence Brooks' Theorem may be applied to $G[X_n]$. Hence,

$$\begin{aligned}\chi(G) &\leq \Sigma_{i=1}^{n} \chi(G[X_i]) \\ &\leq (n-1)(r-1) + \Delta(G) - r(n-1) \\ &= \Delta(G) + 1 - n,\end{aligned}$$

and the theorem is proved.

We know of no examples with $\chi(G) < \Delta(G)$ for which Theorem 5.1 holds with equality.

It has recently come to our attention that O. V. Borodin and A. V. Kostochka have independently obtained Theorem 5.1. Their result appears in a preprint titled "On an Upper Bound of the Graph's Chromatic Number Depending on Graph's Degree and Density."

6. The chromatic number, clique number and maximum degree of a graph.

In this section we obtain results concerning the structure of a graph G having the parameters

$$\Delta(G) = h, \quad \theta(G) = h - r, \quad \chi(G) = h - r + 1,$$

where h and r are integers. Our main concern is with $h \geq 6$ and $r = 1$. The case $r = 0$ is Brooks' Theorem (Theorem 2.1), when $h \geq 3$.

<u>Theorem 6.1</u> Let r and h be integers, where $0 \leq r < h$. Let G be an edge-minimal graph satisfying

(6.1) $\Delta(G) \leq h, \quad \theta(G) \leq h - r, \quad \chi(G) \geq h - r + 1.$

For each $e \in E(G)$ there is a maximal stable set S_e such that either e lies in all cliques K_{h-r} of $G - S_e$, or e lies in an edge-minimal subgraph H of $G - S_e$ satisfying

(6.2) $\Delta(H) \leq h - 1, \quad \theta(H) \leq h - r - 1, \quad \chi(H) = h - r.$

<u>Proof</u>: Assume G to be an edge-minimal graph with

(6.3) $\Delta(G) \leq h,$

(6.4) $\theta(G) \leq h - r,$

(6.5) $\chi(G) \geq h - r + 1.$

The edge-minimality of G implies that for any $e \in E(G)$,

(6.6) $\chi(G - e) = h - r.$

Hence, (6.5) becomes

(6.7) $\chi(G) = h - r + 1.$

By (6.6) and (6.7), for any maximal stable set $S \subseteq V(G)$,

(6.8) $\chi(G - S) = h - r.$

By (6.6), for any $e \in E(G)$, there is a maximal stable set S_e such that S_e is monochromatic in an $(h - r)$-coloring of $G - e$. Therefore,

(6.9) $\chi(G - e - S_e) = h - r - 1,$

and by (6.3) and the maximality of S_e,

(6.10) $\Delta(G - S_e) \leq h - 1,$

and by (6.8),

(6.11) $\chi(G - S_e) = h - r.$

Since (6.11) precludes $\theta(G - S_e) > h - r$, either

(6.12) $\theta(G - S_e) = h - r,$

or

(6.13) $\theta(G - S_e) < h - r.$

If (6.12) holds, then (6.9) implies that e lies in all cliques K_{h-r} of $G - S_e$. If (6.13) holds, then by (6.10), (6.13), and (6.11), $H = G - S_e$ satisfies the relations (6.2). Also, since the removal of e from $G - S_e$ reduces the chromatic number of $G - S_e$, by (6.9), e is in an edge-minimal subgraph H of $G - S_e$ that satisfies (6.2).

Lemma 6.2 Suppose that G is a connected graph with

(6.14) $\Delta(G) = h, \quad \theta(G) = h-r, \quad \chi(G) = h-r+1,$

such that every edge lies in a clique K_{h-r}. If

(6.15) $h \geq 3r+3,$

then every two cliques on $h-r$ vertices intersect in at least $h-2r-1$ vertices.

Proof: Suppose first that two cliques of G intersect at a vertex v. We claim that these two cliques must intersect in at least $h-2r-1$ vertices. Note that v is adjacent to $h-r-1$ vertices in each clique. If these two cliques overlap at v and at most $h-2r-3$ other vertices, then v is adjacent to at least

$$2(h-r-1)-(h-2r-3) = h+1$$

vertices of G, contrary to (6.14). This proves the claim.

Suppose that C_1 and C_0 are cliques on $h-r$ vertices each, which do not overlap. Since G is connected, there is a minimum length path $v_0, v_1, \ldots, v_n$ in G, where $\{v_0, v_1\} \in E(C_1)$ and $\{v_{n-1}, v_n\} \in E(C_0)$, and since C_1 and C_0 do not overlap, $n \geq 3$. We shall find a shorter path with these properties, contrary to the minimality of n.

For $i = 1,2,3$, denote by C_i the clique on $h-r$ vertices containing the edge $\{v_{i-1}, v_i\}$. By hypothesis, and since $n \geq 3$, such cliques exist. By the claim, C_1

and C_2 overlap in at least $h-2r-1$ vertices, as do C_3 and C_2. Since $|V(C_2)| = h-r$, the number of vertices common to C_1, C_2, and C_3 is at least

$$\begin{aligned} |V(C_1 \cap C_2)| + |V(C_3 \cap C_2)| - |V(C_2)| &\geq 2(h-2r-1) - (h-r) \\ &= h-3r-2 \\ &\geq 1, \end{aligned}$$

by (6.15). Let v denote a vertex at which C_1 and C_3 overlap. The path $v_0, v, v_3, \ldots, v_n$ violates the minimality of n. This proves the lemma.

We do not assume Brooks' Theorem in the following:

<u>Theorem 6.3</u> If $h \geq 3$, then there is no graph G with

(6.16) $\Delta(G) = h, \quad \theta(G) = h, \quad \chi(G) = h+1,$

in which each edge of G lies in a clique K_h.

<u>Proof</u>: Suppose that such a graph exists. Let C be a clique K_h. By Lemma 6.2, with $r=0$, each vertex of $G-C$ lies in a clique K_h that intersects C in at least $h-1$ vertices. Hence, each vertex of $G-C$ is adjacent to at least $h-1$ vertices of C. If $|V(G-C)| \geq 2$, then there are at least $2(h-1)$ edges with exactly one end in C.

However, since each vertex of C has degree at most h, and is adjacent to $h-1$ vertices in C, each vertex of C is incident with at most one edge having just one end in H. Thus, there are at most h edges with just one end in C. This contradiction shows that $|V(G-C)| \leq 1$. But since $\theta(G) = h$, this forces $\chi(G) = h$, and hence G does not exist.

Theorem 6.4 If $h \geq 6$, then there is no graph with

(6.17) $\Delta(G) = h, \quad \theta(G) = h-1, \quad \chi(G) = h$

in which each edge of G lies in a clique K_{h-1}.

Proof: Let C be a clique K_{h-1} of G chosen to have at least as many vertices of degree less than h as any other clique.

By Lemma 6.2, with $r = 1$, each vertex of $G-C$ lies in a clique that intersects C in at least $h-3$ vertices. Hence, each vertex of $G-C$ is adjacent to at least $h-3$ vertices of C. Therefore, there are at least $(h-3)|V(G-C)|$ edges with exactly one end in C.

Case I: Suppose that each vertex of C has degree h. By the choice of C, it follows that each vertex of G has degree h. Hence, each vertex of C is adjacent to 2 vertices outside of C, and so there are $2|V(C)| = 2(h-1)$ edges with exactly one end in C. Thus,

$$2(h-1) \geq (h-3)|V(G-C)|,$$

whence,

$$|V(G-C)| \leq 2\frac{h-1}{h-3} \leq \frac{10}{3},$$

since $h \geq 6$. If $|V(G-C)| = 3$, then since each vertex of $V(G-C)$ is adjacent to at most two vertices of $V(G-C)$, each is adjacent to at least $h-2$ vertices of C. This gives at least $(h-2)|V(G-C)|$ edges with exactly one end in C. Thus,

$$2(h-1) \geq (h-2)|V(G-C)|,$$

whence,

$$|V(G-C)| \leq 2\frac{h-1}{h-2} \leq \frac{5}{2}.$$

<u>Case II</u>: Suppose that at least two vertices of C have degree less than h. Hence, the number of edges with exactly one end in C is at most $2(h-2)$. Thus,

$$2(h-2) \geq (h-3)|V(G-C)|,$$

whence

$$|V(G-C)| \leq 2\frac{h-2}{h-3} \leq \frac{8}{3}.$$

<u>Case III:</u> Suppose that exactly one vertex of C has degree less than h. Hence, the number of edges with exactly one end in C is at most $2(h-1)-1 = 2(h-\frac{3}{2})$. Thus,

$$2(h-\tfrac{3}{2}) \geq (h-3)|V(G-C)|,$$

whence,

$$|V(G-C)| \leq \frac{2h-3}{h-3} \leq 3,$$

with equality only if $h=6$ and each vertex of $G-C$ is adjacent to exactly $h-3=3$ vertices of C. In this case, if $v_1 \in V(G-C)$ is adjacent to $h-3=3$ vertices of C, then v_1 is in the same clique K_5 with another vertex $v_2 \in V(G - C)$. By the choice of C, one of v_1, v_2 has degree $h = 6$ in G, for otherwise, we would be in Case II. This vertex is adjacent to at most two other vertices of $V(G - C)$, and hence to four vertices of C. But this contradicts the earlier remark that each vertex of $G - C$ is adjacent to exactly three vertices in C.

Therefore, in any case,

$$|V(G - C)| \leq 2.$$

If $|V(G - C)| \leq 1$, then $|V(G)| \leq h$, and so $\Delta(G) \leq h - 1$ and $\theta(G) = h - 1$ imply $\chi(G) = h - 1 < h$. Thus, we may assume that $|V(G - C)| = 2$ and $|V(G)| = h + 1$. Let S be a maximum stable set in $V(G)$. If $|S| \geq 3$, then $\chi(G) < h$. Since $\theta(G) = h - 1$, $|S| \geq 2$. Suppose, therefore, that $|S| = 2$. Write $S = \{s_1, s_2\}$. If $G - S$ is not a clique K_{h-1}, then $\chi(G - S) \leq h - 2$, whence $\chi(G) < h$. On the other hand, suppose that $G - S$ is a clique K_{h-1}. Since $\theta(G) = h - 1$, s_1 is not adjacent to some vertex $v_1 \in V(G - S)$, and s_2 is not adjacent to some point $v_2 \in V(G - S)$. Since S is a maximum stable set, $v_1 \neq v_2$. Thus, since

$$\chi(G - S - \{v_1, v_2\}) = |V(G - S - \{v_1, v_2\})| = h - 3,$$

and since $\{s_1, v_1\}$ and $\{s_2, v_2\}$ are stable sets, $\chi(G) < h$. Therefore, G does not exist, and the proof of Theorem 6.4 is complete.

Both Theorem 6.3 and 6.4 are best possible in a certain sense. If $h = 2$, then Theorem 6.3 fails for an odd polygon of at least five vertices. Suppose that $h = 5$ in Theorem 6.4. We construct a counterexample G as follows. Let V(G) be a set of $4n + 2$ vertices, $n \geq 2$, and let π map them onto the vertices of a polygon G' on $2n + 1$ vertices so that exactly two vertices of V(G) are mapped to each vertex of G'. We define the edges of G to be the pairs v_1, v_2 such that either $\pi(v_1) = \pi(v_2)$ or $\pi(v_1)$ and $\pi(v_2)$ are adjacent in G'.

<u>Theorem 6.5</u> Let $r = 0$ or 1. If for some $h \geq 3r + 3$ there is a graph G with

(6.18) $\quad \Delta(G) \leq h, \quad \theta(G) \leq h - r, \quad \chi(G) = h - r + 1,$

then there is a subgraph H of G, outside of a maximal stable set S, which is edge-minimal with respect to

(6.19) $\quad \Delta(H) \leq h - 1, \quad \theta(H) \leq h - r - 1, \quad \chi(H) = h - r.$

<u>Proof</u>: Without loss of generality, we may assume that G is edge-minimal with respect to (6.18). By Theorem 6.1, with $r = 0$ or 1, each edge e of G either lies in a clique K_{h-r} of $G - S_e$, for some maximal stable set $S_e \subseteq V(G)$, or there is a subgraph H of $G - S_e$ satisfying (6.19). By Theorems 6.3 and 6.4, it is not possible that each edge $e \in E(G)$ lies in a clique

K_{h-r}, for no such graph exists. Thus, there is an edge e contained in a subgraph H of G satisfying (6.19).

<u>Corollary 6.6</u> If Brooks' Theorem holds for all graphs H with $\Delta(H) = 3$, then Brooks' Theorem holds for all graphs.

<u>Proof</u>: Brooks' Theorem (Theorem 2.1) for $\Delta(H) = 3$ is a basis for induction. By Brooks' Theorem for $\Delta(H) = h-1$, there is no graph satisfying (6.19). Thus, by Theorem 6.5 with $r = 0$, there is no graph G satisfying (6.18), and so Brooks' Theorem holds for $\Delta(G) = h$.

<u>Corollary 6.7</u> If there is an integer $n \geq 6$ such that there is no graph H satisfying

(6.20) $\quad \Delta(H) = n, \quad \theta(H) = n-1, \quad \chi(H) = n,$

then for all $h \geq n$, there is no graph G satisfying

(6.21) $\quad \Delta(G) = h, \quad \theta(G) = h-1, \quad \chi(G) = h.$

<u>Proof</u>: We use the nonexistence of a graph H satisfying (6.20) as a basis for induction. Suppose there is no graph H satisfying

$$\Delta(H) = h-1, \quad \theta(H) = h-2, \quad \chi(H) = h-1$$

where $h \geq 7$. By Theorem 6.5, with $r = 1$, there is no graph G satisfying (6.21).

Benedict and Chinn [2] note that for $n \leq 7$ there are graphs H satisfying (6.20). Thus, the induction suggested by Corollary 6.7 would have to start at $n \geq 8$, if at all.

We show that there are infinitely many graphs G satisfying

$$(6.22) \quad \Delta(G) = 6, \quad \theta(G) = 5, \quad \chi(G) = 6.$$

We define such graphs recursively. Let G' be the graph obtained from K_7 by the removal of three edges that form a triangle in K_7. Let G_0 be either K_6 or a graph that satisfies (6.22). Given G_i, let G_{i+1} be obtained from G_i and G' by removing from G_i a vertex (but not its incident edges) and joining these incident edges to the three vertices of degree four in G' (called vertices of attachment), so that at most two edges from G_i are assigned to each of the three vertices of degree four in G'. Suppose, by way of contradiction, that $\chi(G_{i+1}) = 5$. Since 4 colors are assigned to the 4 vertices of degree 6 in G', a fifth color must be assigned to each of the three vertices of attachment of G'. Hence, in a 5-coloring of G_{i+1}, the 7 vertices of G' behave like a single vertex of the fifth color. Therefore, $\chi(G_{i+1}) = \chi(G_i) = 6$, a contradiction. Since G_0

satisfies $\chi(G_0) = 6$, we have $\chi(G_{i+1}) = 6$, by induction. It is clear that the other relations of (6.22) also hold for G_{i+1}.

We give seven nonisomorphic examples of connected graphs G with

$$\Delta(G) = 7, \quad \theta(G) = 6, \quad \chi(G) = 7.$$

Define the graph G' to be a clique K_8 minus 3 edges which form a triangle in K_8. Thus, G' has 3 vertices of degree 5 and 5 vertices of degree 7. For any nonempty subset S of the set of vertices of a clique K_7, construct G by removing each vertex of S (but not the incident edges) and replacing it with a copy of G' so that the six edges incident with a removed vertex are instead made to be incident in pairs with the 3 vertices of degree 5 in the copy of G'. This gives a graph G having the desired parameters. The number of vertices of G is thus $7(|S|+1)$. Benedict and Chinn obtained the graph with $|S| = 1$ as an example G having these parameters, and noted that the method of construction does not generalize to $n \geq 8$.