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Supereulerian Graphs of Minimum Degree at Least 4

Paul A. Catlin

(Dept. of Math., Wayne State Univ., Detroit MI, 48202, U. S. A)

Li Xiangwen

(Dept. of Math., Central China Normal Univ., Wuhan, Hubei, 430079, P. R. China)

Abstract Let G be a 2-edge-connected graph of order n. Suppose that for any bond $E \subsetneq E(G)$ with $|E| \leq 3$, each component of $G - E$ has order at least $\frac{n}{5}$. Then either G has a spanning closed trail or G has five disjoint connected subgraghs, all of order $\frac{n}{5}$, such that when all five of them are contracted, G is contracted to $K_{2,3}$. Various prior conditions(due to X. T. Cai, P. A. Catlin, F. Jaeger, and H. -J. Lai) for a graph to be supereulerian, i.e., to have a spanning closed trail, follow.

Key words supereulerian graph; collapsibie; reduction; contraction

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1 Introduction

We use [2] for terminology and notation not defined here and consider only finite, undirected graphs, and we allow graphs to have multiple edges but no loops.

A bond of G is a minimal edge set whose removal disconnects G. A contraction of a graph G is any graph obtained from G by contracting a set (possibly empty) of edges and deleting all resulting loops. For a graph G with a connected subgraph H, the contraction G/H is the multigraph obtained from G by replacing H by a vertex v_H, such that the number of edges in G/H joining any $v \in V(G-H)$ to v_H in G/H equals the number of edges joining v in G to $V(H)$. Note that multiple edges can arise in contractions. A graph is eulerian if it is connected and every vetex has even degree (In particular, K_1 is eulerian).

A graph is called *Supereulerian* if it has a spanning closed trail, and K_1 is regarded as supereulerian. For any graph H, define

$$O(H) = \{ \text{ odd-degree vertices of } H\}.$$

By Euler's Theorem, G is supereulerian if and only if G has a connected spanning subgraph G_0 such that $O(G_0) = \emptyset$. Denote by $\mathcal{SL}$ the family of all supereulerian graphs.

In this paper, we give a sufficient condition for a graph G with $\delta(G) \geq 4$ such that $d(u) + d(v) \geq \frac{2n}{5} - 1$ for every edge $uv \in E(G)$ to be supereulerian. As corollaries, it gives various sufficient conditions due to several authors for a graph to be supereulerian.

2 The Reduction Method

A graph G is called *collapsible* if for every even set $X \subseteq V(G)$, there is a spanning connected subgraph G_X of G such that $O(G_X) = X$.

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Note that K_1 is collapsible, since $\emptyset$ is the only even subset of $V(K_1)$. K_2 is not collapsible since there does not exist some spanning connected subgraph H of K_2 such that $O(H) = \emptyset$ which is a even subset of $V(K_2)$. Any collapsible graph G is supereulerian since $\emptyset$ is a even subset of $V(G)$. Denote by $\mathcal{CL}$ the family of all collapsible graphs. It was noted in [6] that $C_3 \in CL$, and that if G has two edge-disjoint spanning trees, then $G \in \mathcal{CL}$.

The following theorem is a corollary of Theorem 3 of [6].

Theorem 1[6] Let H be a subgraph of G. If $H \in \mathcal{CL}$ then

(a) $G \in \mathcal{SL} \Longleftrightarrow G/H \in \mathcal{SL}$; and

(b) $G \in \mathcal{CL} \Longleftrightarrow G/H \in \mathcal{CL}$.

Catlin[7] conjectured that if $H \notin \mathcal{CL}$ then there is a supergraph G of H such that the equivalence (a) of Theorem 1 fails.

Theorem 2 ([6, Theorem 4]) Let H_1 and H_2 be subgraphs of G. If $H_1, H_2 \in \mathcal{CL}$ and $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{CL}$.

We can define a relation on $V(G)$ as follows: for two vertices $u, v \in V(G)$, u is related to v if and only if $u, v \in V(H)$ for collapsible subgraph H of G. Using Theorem 2, we can check that this is an equivalence relation on $V(G)$. The equivalence classes induce the maximal collapsible subgraphs of G.

Since $K_1 \in \mathcal{CL}$, any graph G has a collection $H_1, H_2, \cdots, H_c$ (say) of maximal collapsible subgraphs. It follows from Theorem 2 that the H_i' s are disjoint and uniquely determined. Let G' denote the graph of order c obtained from G by contracting each H_i to a vertex $v_i (1 \le i \le c)$. We call G' the *reduction* of G. Then H_i is called the *preimage* of v_i. If $H_i = K_1$, H_i is called the trivial collapsible subgraph. (In [6], G' was denoted by G_1). Repeated applications of (a) of Theorem 1 give:

Theorem 3 ([6, Theorem 8]) For any graph G,

$$G \in \mathcal{SL} \Longleftrightarrow G' \in \mathcal{SL}.$$

A graph is called reduced if it is the reduction of some graph.

Theorem 4 ([6, Theorem 5]) A graph is reduced if and only if it has no nontrivial collapsible subgraph.

Theorem 5 ([6, Theorem 8 and Lemma 5]) Let G be a nontrivial graph and let

$$V_4 = \{v \in V(G) | d(v) < 4\}. \quad (1)$$

If G is a reduced graph, then each of these holds:

(a) G is a simple graph;

(b) G has no cycle of length less than 4;

(c) If G is 2-edge-connected then either $|V_4| = 4$ and G is eulerian or $|V_4| \ge 5$.

3 The Main Result

Theorem 6 Let G be a 2-edge-connected graph of order n. Suppose, for any bond $E \subseteq E(G)$ with $|E| \le 3$, that each component of $G - E$ has order at least $\frac{n}{5}$. Then exactly one of the following holds:

(1) $G \in \mathcal{SL}$;

(2) G has five disjoint collapsible subgraphs, say $H_1, H_2, \cdots, H_s$, such that each $|V(H_i)| = \frac{n}{5}$ whenever $1 \le i \le 5$; and if each H_i is contracted to a vertex $(1 \le i \le 5)$, then G is contracted to $K_{2,3}$ (see Figure 1).

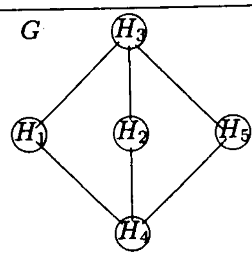

Figure 1

Proof Let $H_1, H_2, \cdots H_c$ be the maximal collapsible subgraphs of G. As already noted, Theorem 2 implies that these H_i's are disjoint and uniquely determined. Since $K_1 \in \mathcal{CL}$, each vertex of G is in some $H_i (1 \leq i \leq c)$. Let G' be the reduction of G (obtained by contracting every H_i to a vertex, say v_i, where $1 \leq i \leq c$).

By Theorem 4, G' is reduced and since G is 2-edge-connected, either $G' = K_1$ or G' is 2-edge-connected and nontivial. Since $K_1 \in \mathcal{SL}$, if $G' = K_1$ then $G \in \mathcal{SL}$ by Theorem 3. Thus, we suppose that G' is 2-edge-connected and nontrivial, and we define

$$V_4 = \{v \in V(G') | \, d_{G'(v)} < 4\}.$$

By (c) of Theorem 5, either $|V_4| = 4$ and $G' \in \mathcal{SL}$ or $|V_4| \geq 5$. In the former case, $G \in \mathcal{SL}$, by Theorem 3. It remains to suppose

$$|V_4| \geq 5. \tag{2}$$

We can assume that $v_1, v_2, \cdots, v_5$ are vertices of $V(G')$ in V_4. Thus, $d(v_i) \leq 3$ for $1 \leq i \leq 5$. The corresponding maximal collapsible subgraphs are $H_1, H_2, \cdots, H_5$. Each $H_i (1 \leq i \leq 5)$ is joined to the rest of G by a bond consisting of the $d(v_i) \leq 3$ edges that are incident in G' with v_i. By the hypothesis of Theorem 6,

$$|V(H_i)| \geq \frac{n}{5}, \tag{3}$$

and so

$$n = |V(G)| \geq \sum_{i=1}^{5} |V(H_i)| \geq n. \tag{4}$$

Equality must hold throughout (4). This forces equality throughout (3), for $1 \leq i \leq 5$, and

$$|V(G')| = c = 5. \tag{5}$$

To complete the proof, it suffices to show that G' (the graph arising from G when all H_i's are contracted) is $K_{2,3}$. Since G is 2-edge-connected, so is G', and each edge of G' lies in a cycle. By (5), G' has order 5. By Theorem 3 and $G \notin \mathcal{SL}$, we have $G' \notin \mathcal{SL}$, and so G' is not a 5-cycle. Since G' is a reduced graph, it is a simple graph and it has no cycle of length less than 4, by (a), (b) of Theorem 5. Thus, G' has no odd cycle, and so each edge must lie in a 4-cycle. Therefore,

$$G' = K_{2,3}.$$

This proves Theorem 6.

4 Special Cases

Various prior results follow from Theorem 6.

Corollary 6A[12] Any 4-edge-connected graph is supereulerian.

Proof If G is 4-edge-connected, then G satisfies the hypothesis of Theorem 6 and is not contractible to $K_{2,3}$.

Corollay 6B Let G be a 2-edge-connected simple graph of order n. If $\delta(G) \geq 4$ and if every edge $uv \in E(G)$ satisfies

$$d(u) + d(v) \geq \frac{2n}{5} - 2, \tag{6}$$

then G satisfies a conclusion of Theorem 6.

Proof Let G be a 2-edge-connected simple graph of order n with $\delta(G) \geq 4$, such that each edge uv of G satisfies (6). It suffices to show that G satisfies the hypothesis of Theorem 6.

Let E be a bond of G with $|E| \leq 3$, and let G_1 and G_2 be the two components of $G - E$, where $|V(G_1)| \leq |V(G_2)|$. It suffices to show that $|V(G_1)| \geq \frac{n}{5}$. For any $e \in E(G)$, let n_e denote the number of edges of the bond E adjacent in G to e.

By $\delta(G) \geq 4$, we have $|V(G_1)| > 1$. Hence, the component G_1 has an edge, say uv. By $\delta(G) \geq 4$ and since G is simple,

$$4 + 4 \leq d(u) + d(v) \leq 2(|V(G_1)| - 1) + n_{uv} \leq 2|V(G_1)| + 1.$$

Hence, $|V(G_1)| \geq 4 > 3 \geq |E|$, and so G_1 has a vertex w (say) that is not incident with any edge of E. By $d(w) \geq \delta(G) \geq 4 > 3 \geq |E|$, w has a neighbor in G_1, say x, that is also not incident with any edge of E. By (6), since G is simple, and by $n_{wx} = 0$,

$$\frac{2n}{5} - 2 \leq d(w) + d(x) \leq 2(|V(G_1)| - 1) + n_{wx} = 2|V(G_1)| - 2.$$

Therefore, $\frac{n}{5} \leq |V(G_1)|$, and so G satisfies the hypothesis of Theorem 6. Hence, G satisfies a conclusion of Theorem 6.

For graphs of minimum degree at least 4, Corollary 6B implies prior results of Benhocine, Clark, Kohler, and Veldman[1,Theorem4], Brualdi and Shanny[3], Catlin[5], Clark[10], and Lai[13,14], as well as the following result:

Corollary 6C[4,6] Let G be a 2-edge-connected simple graph of order n. If

$$\delta(G) \geq \frac{n}{5} - 1$$

and if $n > 20$ then G satisfies the conclusion of Theorem 6, where each H_i is either a complete graph, or one edge short of being complete.

Proof By Corollary 6B.

5 Other Remarks

By remarks of Jaeger[12], any planar graph G with $\delta(G) \geq 4$ is supereulerian.

The following result has recently been proved, but compared to THeorem 6, its proof is long and difficult.

Theorem 7 If a 2-edge-connected graph G is at most two edges short of having two edge-disjoint spanning trees, then either $G \in \mathcal{SL}$ or G is cotractible to $K_{2,t}$ for some odd integer $t \geq 3$.

It can be shown that any graph G satifying the hypothesis of Theorem 6 must satisfy the hypothesis of Theorem 7. Furthemore, if the graph G satisfies the later conclusion of Theorem 7, then it is easily seen that the hypothesis of Theorem 6 forces $t = 3$.

A corollary of Theorem 7 is an earlier result of Catlin[6], that if a graph G is at most one edge short of having two edge disjoint spanning trees, then either $G \in \mathcal{CL} \subset \mathcal{SL}$ or G has a cut-edge, but not both. An earlier consequence of these results is Jaeger's Theorem[12], that if a graph G has two edge-disjoint spanning trees, then $G \in \mathcal{SL}$. Jaeger[12] observed that this result implies Corollary 6A, also due to Jaeger. The hypotheses of both results of Jaeger are related.

Zhan[15] proved that if G is 4-edge-connected, then $G - \{e, e'\}$ has two edge-disjoint spanning trees, for any $e, e' \in E(G)$. Catlin[7] noted that the converse holds, and more generally, that for any $k \in N$, a graph G is $2k$-edge-connected if and only if $G - E_k$ has k edge-disjoint spanning trees, for any k-element subset $E_k \subseteq E(G)$.

Z. H. Chen[9] has obtained the analogue of Theorem 6 for 3-edge-connected graphs.

Theorem 8[9] Let G be a 3-edge-connected graph of order n. If every bond $E \subseteq E(G)$ with $|E| = 3$ satisfies the property that each component of $G - E$ has order at least $\frac{n}{10}$, then exactly one of the following holds:

(1) $G \in \mathcal{SL}$;

(2) G has ten disjoint collapsible subgraphs $H_1, H_2, \cdots, H_{10}$ such that $|V(H_i)| = \dfrac{n}{10}(1 \le i \le 10)$; and when every H_i is contracted to a vertex, G is contracted to the Petersen graph.

When proving Theorem 8, Chen[9] showed that the only 3-edge-connected simple graph of order at most 11 that is not collapsible is the Petersen graph. There is a 3-edge-connected simple graph G of order 12 that is not collapsible: it has a triangle H such that G/H is the Petersen graph.

Benhocine, Clark, Kohler and Veldman[1] conjectured that if a graph of sufficiently large order n satisfies the condition that

$$d(u) + d(v) > \frac{n}{5} - 2$$

for every edge $uv \in E(G)$, then the line graph of G is hamiltonian. By the characterization of hamiltonian line graphs due to Harary and Nash-Williams[11] , if $G \in \mathcal{SL}$ then G has a hamiltonian line graph. Therefore Corollary 6B implies the Benhocine-Clark-Kohler-Veldman conjecture for graphs with minimum degree at least 4, because G is supereulerian in Corollary 6B if (6) is strict.

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