A Reduction Criterion for Supereulerian Graphs*

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ABSTRACT

Let G be a graph, and let H be a connected subgraph of G. When it is known that the graph G/H (obtained from G by contracting H to a vertex) has a spanning eulerian subgraph, under what conditions can it be inferred that G itself has a spanning eulerian subgraph? © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

We follow the notation of Bondy and Murty [1], but graphs have no loops.

Let H be a connected subgraph of G. The *contraction* G/H is the graph obtained from G by contracting H to a vertex and by deleting any resulting loops. A graph is *eulerian* if it is connected, and if all of its vertices have even degree. A graph is *supereulerian* if it has a spanning eulerian subgraph. Denote

$$\mathcal{SL} = \{\text{supereulerian graphs}\}.$$

Extensive literature on $\mathcal{SL}$ is surveyed in [3] and [4].

Call a graph H *collapsible* if for every even set $X \subseteq V(H)$, there is a spanning connected subgraph H_X of H such that X is the set of vertices with odd degree in H_X. Denote

$$\mathcal{CL} = \{\text{collapsible graphs}\}.$$

For example, K_1 and cycles of length less than 4 are collapsible, but C_4 is not. We proved [2] that $H \in \mathcal{CL}$ if H has two edge-disjoint spanning trees.

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The fundamental importance of $\mathcal{CL}$ in studying $\mathcal{SL}$ arises from the following reduction theorem:

Theorem 1 (Catlin [2]). Let H be a connected subgraph of G. If $H \in \mathcal{CL}$, then

$$G \in \mathcal{SL} \Leftrightarrow G/H \in \mathcal{SL}. \tag{1}$$

We conjectured (see [3]) that if $H \notin \mathcal{CL}$ then (1) fails for some supergraph G of H. Below we give a criterion on H for (1) to hold for all supergraphs G of H. Also, we show that for certain supergraphs G of H, (1) can hold even when H is not collapsible.

2. THE MAIN RESULTS

It is easy to show that if $G \in \mathcal{SL}$ then any contraction of G is in $\mathcal{SL}$, i.e., $\mathcal{SL}$ is closed under contraction.

Definition. Let S be a set. A *pairing* A in S is a family of mutually disjoint 2-subsets of S. A pairing may or may not be a partition of S.

Definition. Let H be a graph. For any pairing

$$A = \{\{v_1, v_1''\}, \{v_2, v_2''\}, \ldots, \{v_k, v_k''\}\}$$

of k (say) disjoint pairs of vertices in $V(H)$, let $H(A)$ denote the supergraph of H obtained by adding to H the k paths $P_1, P_2, \ldots, P_k$ such that both

(i) P_j is the path $v_j v_j' v_j'' (1 \le j \le k)$; and
(ii) $v_j' \notin V(H) (1 \le j \le k)$.

Theorem 2. Let G be a graph, and let H be a connected proper subgraph of G. Let $G_1, G_2, \ldots, G_p$ denote the components of $G - E(H)$ having at least one vertex not in $V(H)$. If $H(A) \in \mathcal{SL}$ for every pairing A in $V(H)$ satisfying

$$\{v_i, v_i''\} \in A, v_i \in V(G_s), v_i'' \in V(G_t) \Rightarrow s = t, \tag{2}$$

then

$$G \in \mathcal{SL} \Leftrightarrow G/H \in \mathcal{SL}. \tag{3}$$

Proof. Let G be a graph, let H be a subgraph of G, and let $G_1, G_2, \ldots, G_p$ denote the p (say) components of $G - E(H)$ that contain some vertex outside of $V(H)$. Suppose $H(A) \in \mathcal{SL}$ for every pairing A in $V(H)$ satisfying (2). Since $\mathcal{SL}$ is closed under contraction (see above), G/H is in $\mathcal{SL}$ if G is in $\mathcal{SL}$.

Suppose $G/H \in \mathcal{SL}$. Then G/H has a spanning closed trail T. Viewing T as an edge sequence of G we observe that $T \cap E(H) = \emptyset$ and that T is an edge-disjoint union of r (say) closed trails $T_j^C (1 \le j \le r)$ and m (say) maximal open trails $T_j (1 \le j \le m)$ whose ends are in $V(H)$. Observe that every T_j^C has also a vertex in common with $V(H)$ if one assumes w.l.o.g. that these T_j^C correspond to the eulerian components of T in $G - E(H)$. Denote the distinct ends of T_j by $\{v_j, v_j''\}$. Since the open trails are maximal, the m sets $\{v_j, v_j''\} (1 \le j \le m)$ are disjoint in $V(H)$. Denote

$$A = \{\{v_1, v_1''\}, \{v_2, v_2''\}, \ldots, \{v_m, v_m''\}\}.$$

Note that A is a pairing in $V(H)$ satisfying (2). Hence $H(A) \in \mathcal{SL}$, and so there is a spanning closed trail T_A (say) in $H(A)$. Define

$$E_0 = E(T_A[V(H)]) \cup \bigcup_{j=1}^{m} E(T_j) \cup \bigcup_{j=1}^{r} E(T_j^C).$$

Then $E_0 \subseteq E(G)$, and the graph $G[E_0]$ is the desired spanning eulerian subgraph of G. For $G[E_0]$ is an even subgraph by construction of A and the definition of $H(A), T_A[V(H)]$, respectively; it is connected because it can be viewed as being obtained from T_A by replacing each P_i (see the definition of $H(A)$) with (the connected) T_i, and attaching the T_j^C to some vertex x_j in $V(H)$; and it is spanning by definition of E_0. This proves (3). ∎

Here is an easy illustration of Theorem 2. Let G be a graph with a 4-cycle H given by $wxyzw$, and suppose the *special* case that $G - \{wx, yz\}$ is disconnected. Then the pairings A satisfying (2) must satisfy

$$A \subseteq \{\{w, z\}, \{x, y\}\}.$$

It is easily checked that for any such pairing $A, H(A) \in \mathcal{SL}$. Hence, (3) holds. However in the *general* case when H is a 4-cycle in G, (3) can fail since C_4 is not collapsible (see Theorem 1).

Theorem 3. Let H be a connected graph. The equivalence

$$G \in \mathcal{SL} \Leftrightarrow G/H \in \mathcal{SL} \tag{4}$$

holds for every supergraph G of H, if and only if $H(A) \in \mathcal{SL}$ for every pairing A in $V(H)$.

Proof. Let H be a connected graph. Suppose first that (4) holds for every supergraph G of H. For any pairing A in $V(H), H(A)$ is a supergraph of H, and so (4) gives

$$H(A) \in \mathcal{SL} \Leftrightarrow H(A)/H \in \mathcal{SL}.$$

Since $H(A)/H$ is obviously supereulerian (every vertex has degree 2, except possibly the one to which H is contracted), $H(A) \in \mathcal{SL}$, too.

Suppose conversely that $H(A) \in \mathcal{SL}$ for every pairing A in $V(H)$. By Theorem 2, (4) holds. ∎

Theorem 1 is a corollary of Theorem 3. If $H \in \mathcal{SL}$, then it follows from the definition of $\mathcal{CL}$ that $H(A) \in \mathcal{SL}$ for any pairing A in $V(H)$. Hence, (4) and (1) hold.

References

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