Edge-connectivity and edge-disjoint spanning trees

Paul A. Catlin, Department of Mathematics
Wayne State University, Detroit MI 48202

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Abstract

We characterize the edge-connectivity of a graph in terms of its spanning trees. One example: for $k \in \mathbf{N}$, the graph G is $2k$-edge-connected if and only if $G - E$ has k edge-disjoint spanning trees, for any set E of k edges of G.

We use the notation of Bondy and Murty [1]. A well-known characterization of the edge-connectivity of a graph is a variant of Menger's Theorem (see [1]): a graph G is k edge-connected if and only if for any distinct vertices u and v, there are k edge-disjoint paths in G joining u and v. Mader [6] provided a reduction that preserves edge-connectivity. Here we provide a characterization of edge-connectivity in terms of the spanning trees of the graph.

Denote the edge-toughness of a graph G by

$$\eta(G) = \min_{E \subset E(G)} \frac{|E|}{\omega(G - E) - 1}, \tag{1}$$

where $G \neq K_1$ and the minimum in (1) runs over all subsets $E \subset E(G)$ such that $\omega(G - E)$, the number of components of $G - E$, is at least 2. The case $q = 1$ of the following result is a well-known theorem of Tutte [9] and Nash-Williams [7] (characterizing the maximum number of edge-disjoint spanning trees in a given graph as $\lfloor \eta(G) \rfloor$), and the general case of the following result was obtained by Cunningham [3] and extended to matroids by Catlin, Grossman, Hobbs, and Lai [2]:

Theorem 1 [3, 2] Let G be a nontrivial graph and let $p, q \in \mathbf{N}$. Then $\eta(G) \geq p/q$ if and only if G has p spanning trees (repetitions allowed) such that each edge of G lies in at most q of the p trees.

Thus, in the theorem below, the statement $\eta(G - E') \geq k$ is equivalent to the assertion that $G - E'$ has at least k edge-disjoint spanning trees.

Theorem 2 Let G be a nontrivial graph, let k be an integer at most $|E(G)|$, and let $\mathcal{E}_k$ be the collection of all k-element subsets of $E(G)$. Then

$$\kappa'(G) \geq 2k \iff \forall E' \in \mathcal{E}_k,\ \eta(G - E') \geq k. \tag{2}$$

Proof: ($\Longrightarrow$). Suppose that $\kappa'(G) \geq 2k$ and suppose $E \subseteq E(G)$ satisfies $\omega(G - E) \geq 2$. Count the number t (say) of incidences of edges of E. Each component of $G - E$ is incident with at least $\kappa'(G) \geq 2k$ edges of E. Since there are $\omega(G - E)$ components and each edge of E is counted twice,

$$2\,|E| \;=\; t \geq \kappa'(G)\omega(G - E) \;\geq 2k\omega(G - E). \tag{3}$$

Since (3) holds for all edge sets of the form $E \;=\; E' \cup E''$, where

$$E' \in \mathcal{E}_k,\ E' \cap E'' = \emptyset,\ \omega(G - E) \geq 2,$$

it follows that

$$|E''| + k = |E' \cup E''| = |E| \geq k\omega(G - E) = k\omega((G - E') - E''),$$

and so

$$|E''| \geq k[\omega((G - E') - E'') - 1]. \tag{4}$$

Since $E \subseteq E(G)$ is arbitrary with $E' \subseteq E$ and $\omega(G - E) \geq 2$, E'' runs over all subsets of $E(G - E')$ for all $E' \in \mathcal{E}_k$, where $\omega((G - E') - E'') \geq 2$. By (4),

$$\eta(G - E') = \min_{E'' \subseteq E(G - E')} \frac{|E''|}{\omega((G - E') - E'') - 1} \geq k,$$

for all $E' \in \mathcal{E}_k$, and so the right side of (2) holds.

($\Longrightarrow$). Suppose that the left side of (2) is false; i.e., that $\kappa'(G) < 2k$. By the definition of $\kappa'(G)$, there is a set E of fewer than $2k$ edges of $E(G)$, such that $\omega(G - E) \geq 2$. Since $k \leq |E(G)|$, by hypothesis, $\mathcal{E}_k \neq \emptyset$. Let E' be an element of E_k such that either $E' \subseteq E$ (if $|E| \geq k$) or $E \subseteq E'$ (if $|E| < k$). If $E' \subseteq E$, then

$$|E - E'| < k, \tag{5}$$

and

$$2 \leq \omega(G - E) = \omega((G - E') - (E - E')). \tag{6}$$

Therefore, by the definition of $\eta(G - E')$ and by (5) and (6),

$$\eta(G - E') \leq \frac{|E - E'|}{\omega((G - E') - (E - E'))} < |E - E'| < k,$$

and so the right side of (40) is false. If $E \subseteq E'$ instead, then the right side of (2) fails again, because $\omega(G - E) \geq 2$ and hence $\eta(G - E') = 0$. $\Box$

<u>Corollary 3</u> [8, 5, 4] If a graph G is $2k$-edge-connected, for some $k \in \mathbf{N}$, then G has k edge-disjoint spanning trees. $\Box$

<u>Corollary 4</u> [10] If G is 4-edge-connected, then for any $e, e' \in E(G)$, $G - \{e, e'\}$ has two edge-disjoint spanning trees. $\Box$

Theorem 2 asserts both Corollary 4 (due to Zhan) and its converse.

For odd values of the edge-connectivity, there is an analogue of Theorem 2, presented last. Since the proof is analogous, we omit it.

<u>Theorem 5</u> Let G be a nontrivial graph, let k be an integer at most $|E(G)|$, and let $\mathcal{E}_k$ be the collection of k-element subsets of $E(G)$. Then

$$\kappa'(G) \geq 2k + 1 \iff \forall E' \in \mathcal{E}_k, \eta(G - E') > k. \tag{7}$$

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