# On the edge arboricity of a random graph

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**Abstract**

The edge arboricity $a(G)$ of a graph $G$ is the minimum number of acyclic subgraphs whose union covers the edge set of $G$. In this note we show that if the edge probability is given by $p^3 n = c \log n$, then almost every graph has

$$a(G) = \left\lceil \frac{|E(G)|}{n-1} \right\rceil$$

provided the constant $c$ is sufficiently large.

**Dedicated to Roger Entringer on the occasion of his 60th birthday**

## 1 Introduction

The edge arboricity $a(G)$ of a graph $G$ is minimum number of acyclic subgraphs whose union covers the edge set of $G$. Nash-Williams [Na64] proved that

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil \tag{1.1}$$

where the maximum runs over all non-trivial induced subgraphs $H$ of $G$. The first two authors showed [CaC91] that when the edge probability $p$ is fixed, almost all graphs $G$ have the property that $|E(H)|/(|V(H)|-1)$ attains its maximum in (1.1) if and only if $G = H$. Following closely the method of [CaC91], we will extend that result for $p = p(n) \to 0$.

Our sample space consists of all labeled graphs $G$ with $n$ vertices. The vertex set of $G$ is $V(G) = \{1, 2, \ldots, n\}$ and the edge set is $E(G)$. Given the edge probability $0 < p < 1$, the probability of a graph $G$ with $M$ edges is defined by

$$P(G) = p^M (1-p)^{N-M} \tag{1.2}$$

where $N = \binom{n}{2}$, the number of slots available for edges. Thus the sample space consists of Bernoulli trials and the edges are selected independently with probability $p$. Suppose $\mathcal{Q}$ is a set of graphs of order $n$ with some specified property $Q$. If the probability $P(\mathcal{Q})$ approaches 1 as $n$ goes to infinity, then we say that *almost all graphs have property $Q$* or *the random graph has property $Q$ a.s.* (almost surely).

For background material and notation not provided here one can consult the introductory book on random graphs [Pa85] and for the strongest and many of the most recent results we use the extensive and comprehensive treatise [Bo85].

## 2 Edge arboricity

For any non-trivial, connected graph $G$ of order $n$, define

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}, \tag{2.1}$$

where the maximum is taken over all non-trivial subgraphs $H$ of $G$. We use the following elementary inequalities frequently:

$$\frac{|E(G)|}{n-1} \leq \gamma(G) \leq \lceil \gamma(G) \rceil \leq a(G). \tag{2.2}$$

Let $\mathcal{F}(G)$ be the family of non-trivial subgraphs $H$ of $G$ such that

$$\gamma(G) = \frac{|E(H)|}{|V(H)|-1}. \tag{2.3}$$

Thus these graphs $H$ achieve the maximum value in (2.1) and it is also easy to see that $\gamma(H) = \gamma(G)$.

**Theorem 2.1.** With edge probability defined by $p^3 n = c \log n$, if the constant $c$ is at least 28, almost surely $\mathcal{F}(G) = \{G\}$ and hence the edge arboricity is

$$a(G) = \left\lceil \frac{|E(G)|}{n-1} \right\rceil. \tag{2.4}$$

Proof: Suppose $G$ is any connected graph of order $n > 1$. Let $H$ be in the family $\mathcal{F}(G)$ and set $r = |V(H)|$. First we find a lower bound for $r$ in terms of the number of edges of $G$. By the definition of $H$ we have

$$\gamma(H) = \gamma(G) = \frac{|E(H)|}{r-1}. \tag{2.5}$$

Now we combine (2.2) and (2.5) and use the fact that $H$ has order $r$ to obtain .

$$r = \frac{2}{r-1}\binom{r}{2} \geq \frac{2}{r-1}|E(H)| = 2\gamma(G) \geq 2\frac{|E(G)|}{n-1}. \tag{2.6}$$

Next we can use Chebyshev's inequality to derive an approximation for the number of edges in a random graph $G$ from which we can determine a lower bound for $|E(G)|$. See, for example, a special case in exercise 3.1.2 of [Pa85]. For a slightly more general result, we have the following. For any positive sequence $\varepsilon_n \to 0$,

$$|E(G)| \geq p\binom{n}{2}(1-\varepsilon_n), \tag{2.7}$$

provided that $\varepsilon_n^2 p n^2 \to \infty$.

By hypothesis our edge probability is well beyond the threshold for connectedness (see [Bo85] or [Pa85]) so we can assume that almost all graphs are connected.

Combining (2.6) and (2.7) we observe that for almost all graphs, the number $r$ of vertices in a graph $H$ from the family $\mathcal{F}(G)$ satisfies

$$r \geq pn(1-\varepsilon_n), \tag{2.8}$$

provided that the condition in (2.7) on $\varepsilon_n$ is satisfied.

At this point we need an estimate for the number of edges in $H$. Using Theorem 8, p. 44 of [Bo85], we can conclude that $|E(H)|$ is almost surely quite close to $p\,\binom{r}{2}$. In particular, we can conclude that

$$\gamma(H) = \frac{|E(H)|}{r-1} \leq \frac{r}{2}\left\{p + \left(\frac{7p\log n}{r}\right)^{1/2}\right\}, \tag{2.9}$$

almost surely, provided that

$$r \geq (252/p)\log n. \tag{2.10}$$

And this latter condition will be met if the lower bound in (2.8) exceeds the right side of (2.10), i.e. we just need

$$pn(1-\varepsilon_n) \geq (252/p)\log n. \tag{2.11}$$

On solving this equation for $p$, we find that all required conditions on $p$ are met if $p$ is defined as in the hypothesis.

Now we are ready to compare $n$ and $r$ by using the lower bound on $\gamma(G)$ in (2.6) and (2.7) and the upper bound on $\gamma(H)$ from (2.9). Since $\gamma(H) = \gamma(G)$, we have

$$\frac{r}{2}\left\{p + \left(\frac{7p\log n}{r}\right)^{1/2}\right\} \geq \frac{pn}{2}(1-\varepsilon_n). \tag{2.12}$$

On substituting the expression from the hypothesis for $p$ in this inequality, after a few steps we find that

$$n - r \leq c_0(n\log n/p)^{1/2}, \tag{2.13}$$

for suitable $\varepsilon_n$ and where $c_0$ is a constant greater than $\sqrt{7}$.

Now suppose that there is a vertex $v$ of $G$ that is not also in $H$. We are going to find an upper bound for the degree of $v$ in $G$ that is too far from the mean to hold for almost all graphs. This will imply that such vertices almost surely do not exist. Define $H_v$ to be the subgraph of $G$ induced by $v$ together with the $r$ vertices of $H$. By the definition in (2.1),

$$|E(H_v)| \leq \gamma(H_v)r. \tag{2.14}$$

But since $H$ achieves the maximum value in (2.1),

$$\gamma(H_v) \leq \gamma(H). \tag{2.15}$$

Combining (2.14), (2.15) and (2.5), we have

$$|E(H_v)| \le \gamma(H_v)r \le \gamma(H)r = |E(H)| + \gamma(H). \tag{2.16}$$

This implies that the degree of $v$ in $H$ is at most $\gamma(H)$, and hence the degree of $v$ in $G$ is at most $n - r + \gamma(H)$, i.e., almost surely

$$\deg_G v \le n - r + \gamma(H). \tag{2.17}$$

Using the bounds in (2.9) and (2.13), we find

$$\deg_G v \le c_0(n \log n/p)^{1/2} + \frac{r}{2}\left\{p + \left(\frac{7p\log n}{r}\right)^{1/2}\right\}. \tag{2.18}$$

And after a bit of work on the right side of (2.18) in which the value of $c_0$ depends on $c > 28$, we have

$$\deg_G v \le (1-\varepsilon)pn, \tag{2.19}$$

for large $n$ and sufficiently small $\varepsilon > 0$.

This last inequality contradicts a theorem of Erdös and Rényi which states that if $pn/\log n \to \infty$, then almost surely the degrees of all vertices satisfy

$$(1-\varepsilon)pn < \deg_G v < (1+\varepsilon)pn. \tag{2.20}$$

where $\varepsilon > 0$ is arbitrary (see p. 66 of [Pa85]). //

We suspect that the theorem gives the right value for the edge arboricity for much lower edge probabilities but the family $\mathcal{F}(G)$ may not consist of $G$ alone.

# 3 Tree packing number

The *tree packing number* $t(G)$ of a connected graph $G$ is the maximum number of edge-disjoint spanning trees contained in $G$. It can be used as a measure of network vulnerability and is closely related to the edge arboricity $a(G)$. And the same method of [CaC91] can be applied here with the same result. Tutte [Tu61] and Nash-Williams [Na61] proved that

$$t(G) = \lfloor \eta(G) \rfloor, \tag{3.1}$$

where

$$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{c(G-E)-1} \tag{3.2}$$

and $c(G-E)$ is the number of components of $G-E$.

For any graph satisfying $\mathcal{F}(G) = \{G\}$, we always have $\gamma(G) = \eta(G)$ (see [CaGHL92]). But it can be shown that almost surely $\gamma(G)$ is not an integer and hence random graphs for which $\mathcal{F}(G) = \{G\}$, have

$$a(G) = t(G) + 1. \tag{3.3}$$

Of course, we only have found the values of these packing and covering numbers for random graphs when $p$ is defined as in Theorem 2.1.

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