Nonsupereulerian Graphs with Large Size

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Abstract

We study the structure of 2-edge-connected simple graphs with many edges that have no spanning closed trail. X. T. Cai [2] conjectured that any 3-edge-connected simple graph G of order n has a spanning closed trail, if

$$|E(G)| \geq \binom{n-9}{2} + 16.$$

This bound is best-possible. We prove this conjecture, and we obtain a stronger conclusion.

1. INTRODUCTION

We follow the notation of Bondy and Murty [1], except that graphs have no loops, the graph of order 2 and size 2 is called a 2-cycle and denoted C_2, and K_1 is regarded as having infinite edge-connectivity. For a graph G, let $O(G)$ denote the set of vertices of odd degree in G. The set of natural numbers is denoted $\mathbf{N}$. Let $D_1(G)$ denote the set of vertices of degree 1 in G.

A graph G is called supereulerian if it has a spanning connected subgraph H whose vertices have even degree. A graph G is called collapsible if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. Thus, the trivial graph K_1 is both supereulerian and collapsible. Denote the family of supereulerian graphs by $\mathcal{SL}$, and denote the family of collapsible graphs by $\mathcal{CL}$. Obviously, $\mathcal{CL} \subset \mathcal{SL}$, and collapsible graphs are 2-edge-connected. Examples of graphs in $\mathcal{CL}$ include the cycles C_2, C_3, but not C_t if $t \geq 4$.

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Cai [2] conjectured that any 3-edge-connected simple graph G of order n with

$$|E(G)| \geq \binom{n-9}{2} + 16$$

is supereulerian. We shall show that any such graph is collapsible. The Petersen graph is one of infinitely many graphs that show that this inequality is best-possible.

2. THE REDUCTION METHOD

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Even when G is simple, G/H may not be.

Theorem A (Catlin [3]) Let H be a subgraph of G. If $H \in \mathcal{CL}$ then

$$G \in \mathcal{SL} \iff G/H \in \mathcal{SL},$$

and

$$G \in \mathcal{CL} \iff G/H \in \mathcal{CL}. \square$$

In [3] it was shown that if H_1 and H_2 are both collapsible subgraphs of G with at least one common vertex, then $G[V(H_1) \cup V(H_2)] \in \mathcal{CL}$. Thus, any collapsible subgraph of G is contained in a unique maximal collapsible subgraph. For a graph G where H_1, H_2, ..., H_c are all the maximal collapsible subgraphs of G, define G' to be the graph obtained from G by contracting each H_i $(1 \leq i \leq c)$ to a distinct vertex. Since $V(G) = V(H_1) \cup .... \cup V(H_c)$, the graph G' has order c. We call the graph G' the reduction of G, and we call a graph reduced if it is the reduction of some graph. Any graph G has a unique reduction G' [3]. A graph is collapsible if and only if its reduction is K_1.

Let G be a graph. The arboricity of G, denoted $a(G)$, is the minimum number of forests whose union contains $E(G)$. Let $F(G)$ denote the minimum number of edges that must be added to G, to obtain a spanning supergraph containing two edge-disjoint spanning trees.

Theorem B Let G be a graph and let G' be the reduction of G. Then
(a) $G \in \mathcal{SL} \iff G' \in \mathcal{SL}$;
(b) G' is simple, G' has no 3-cycle, and $a(G') \leq 2$;
(c) $K_{3,3} - e$ ($K_{3,3}$ minus an edge) is collapsible;
(d) If $F(G) \leq 1$ then $G' \in \{K_1, K_2\}$;
(e) $G = G'$ if and only if G has no nontrivial collapsible subgraph;
(f) If $a(G) \leq 2$ then

$$|E(G)| + F(G) = 2|V(G)| - 2. \square$$

Parts (a), (b), (d) and (e) of Theorem B are proved in [3] and part (c) was proved in [4]. Part (f) is easy. A characterization of $F(G)$ appears in [6].

3. A GENERAL RESULT

Theorem 1 Let G be a 2-edge-connected simple graph of order n and let $p \in \mathbf{N} - \{1\}$. If

$$|E(G)| \geq \binom{n-p+1}{2} + 2p - 4, \tag{1}$$

then exactly one of these holds:

(a) The reduction of G has order less than p;

(b) Equality holds in (1), G has a complete subgraph H of order $n-p+1$, and the reduction of G is $G' = G/H$, a graph of order p and size $2p-4$;

(c) G is a reduced graph such that either

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{p+1, p+2\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = p+3.$$

Proof: The conclusions (a), (b), and (c) are clearly mutually exclusive.

Fix a reduced graph G_0, and suppose that G is a simple graph of order n with $G' = G_0$. Any 2-edge-connected graph G arises in this manner, for some value of G_0. Denote

$$V(G_0) = \{v_1, v_2, ..., v_c\},$$

and for each $1 \leq i \leq c$, let H_i denote the collapsible subgraph of G contracted to v_i by the reduction-contraction $G \longrightarrow G_0$. If $|E(G)|$ were maximum among all simple graphs G of order n with $G' = G_0$, then at most one H_i $(1 \leq i \leq c)$ is a nontrivial subgraph of G, and this H_i is a complete subgraph of order $n-c+1$. Therefore,

$$|E(G)| \leq |E(H_i)| + |E(G_0)| \leq \binom{n-c+1}{2} + |E(G_0)|, \tag{2}$$

with equality only if G has at most one nontrivial collapsible subgraph H_i and it is a complete subgraph of order $n-c+1$.

If $G_0 = K_1$, then (a) holds, since $p \geq 2$. Thus, we can suppose that $G_0 \neq K_1$. Since G is 2-edge-connected, so is its contraction G_0, and so $G_0 \neq K_2$. Hence by part (d) of Theorem B, $F(G_0) \geq 2$. By (b) of Theorem B, $a(G_0) \leq 2$, and so (f) of Theorem B gives

$$|E(G_0)| \leq 2c - 4. \tag{3}$$

By (2) and (3),

$$|E(G)| \leq \binom{n-c+1}{2} + 2c - 4,$$

with strict equality only if (2) or (3) holds strictly. This and the hypothesis of Theorem 1 give

$$\binom{n-p+1}{2} + 2p-4 \le |E(G)| \le \binom{n-c+1}{2} + 2c-4. \tag{4}$$

Simplification of (4) yields

$$2n(c-p) \le (c-p)(c+p+3) \tag{5}$$

Case 1 Suppose that $c = p$. Then equality holds throughout (4). This equality in (4) forces equality in (3) and in (2). Thus, (b) of Theorem 1 holds.

Case 2 Suppose that $c < p$. Then (a) of Theorem 1 holds.

Case 3 Suppose that $c > p$. This and (5) give

$$2n \le c+p+3. \tag{6}$$

By the definition of c, $c \le n$.

Subcase 3A Suppose that $c = n$. This and the hypothesis of Case 3 imply $p < n$, and so (6) and $c = n$ imply

$$p < n \le p+3. \tag{7}$$

Since $|V(G_0)| = c = n$, it follows that G is reduced, and so $G = G_0$. Hence by (3), $|E(G)| \le 2n-4$. To prove (c) of Theorem 1, it only remains to prove the appropriate lower bound on $|E(G)|$. If $n = p+1$, then (1) gives

$$|E(G)| \ge \binom{2}{2} + 2p-4 = 2p-3 = 2n-5.$$

If $n = p+2$, then (1) gives

$$|E(G)| \ge \binom{3}{2} + 2p-4 = 2p-1 = 2n-5.$$

If $n = p+3$, then

$$|E(G)| \ge \binom{4}{2} + 2p-4 = 2p+2 = 2n-4.$$

By (7), all cases have been considered.

Subcase 3B Suppose $c < n$. By the relations on c and by (6),

$$p < c < n < p+3. \tag{8}$$

Since each term of (8) is an integer,

$$c = p + 1; \quad n = p + 2. \tag{9}$$

But since G is a simple graph of order n, its reduction cannot have order $n-1$. By (9), $|V(G_0)| = c = n - 1$, and so the reduction of G cannot be G_0. This contradicts the definition of G_0 and G, and so Subcase 3B is impossible. □

4. THE REDUCTION OF 4-CYCLES

Suppose that a graph G contains a 4-cycle H. The subgraph H is not collapsible, and the equivalences of Theorem A do not apply in this case, if H is an induced subgraph. However, the theorem below provides an extension of the reduction method to subgraphs that are 4-cycles.

Let G be a graph containing an induced 4-cycle $xyzwx$, and define

$$E = \{xy, yz, zw, wx\}.$$

Define G/π to be the graph obtained from $G - E$ by identifying x and z to form a vertex v_1, by identifying w and y to form a vertex v_2, and by adding an edge v_1v_2.

Theorem C (Catlin [4, p. 241]) For the graphs G and G/π defined above, the following hold:
(a) If $G/\pi \in \mathcal{CL}$ then $G \in \mathcal{CL}$;
(b) $|V(G)| = |V(G/\pi)| + 2$;
(c) $|E(G)| = |E(G/\pi)| + 3$;
(d) If $G/\pi \in \mathcal{SL}$ then $G \in \mathcal{SL}$. □

5. SOME LEMMAS

Lemma 1 (Chen [7]) Let G be a simple 2-edge-connected graph of order at most 7. If G has at most two vertices of degree 2, then $G \in \mathcal{CL}$. □

Lemma 2 (Lai [8]) Let G be a simple connected graph of order at most 11. If $\delta(G) \geq 3$ then either G is the Petersen graph or the reduction of G is K_1 or K_2. □

Chen [7] had first proved Lemma 2 with the stronger hypothesis that $\kappa'(G) \geq 3$.

Lemma 3 Let G be a simple 2-edge-connected graph of order at most 8, and let $u \in V(G)$. If u is the only vertex of degree 2 in G, then $G \in \mathcal{CL}$.

Proof: Let G and u satisfy the hypothesis of Lemma 3. Then $G - u$ is connected. If $\kappa'(G-u) \geq 2$, then use Lemma 1 to see that $G - u \in \mathcal{CL}$. Then $G \in \mathcal{CL}$ follows. If $\kappa'(G-u) < 2$ then $G - u$ has a cut edge e such that some component, say H, of $G - u - e$ has no cut edge. Since u is the only vertex of degree 2 in G, H is nontrivial and H satisfies the hypothesis of Lemma 1 (with H in place of G of Lemma 1). Therefore, H is a nontrivial collapsible subgraph of G. Note that G/H also satisfies the hypothesis of Lemma 1 (with G/H in place of G of Lemma 1), and hence $G/H \in \mathcal{CL}$. By Theorem A, $G \in \mathcal{CL}$. □

Lemma 4 Any 3-edge-connected reduced graph of order 12 is 3-regular.

Proof: Let G be a 3-edge-connected reduced graph of order 12. By (e) of Theorem B,

(10) G has no nontrivial collapsible subgraph.

By way of contradiction, suppose that

(11) G is not 3-regular.

Then G has a vertex x with $d(x) \geq 4$. Since G is reduced, G is simple and has no 3-cycle, by (b) of Theorem B.

We claim

(12) x lies on a 4-cycle.

Suppose not. Since $d(x) \geq 4$ and $\delta(G) \geq 3$, at least 4 paths in G with origin x have length 1, and at least 8 paths with origin x have length 2. Since G has no 2-cycle and no 3-cycle, and since x is in no 4-cycle, no two of these 12 paths have the same terminus. Hence, $|V(G-x)| \geq 12$, a contradiction that proves (12).

By (12), x lies on a 4-cycle, say $xyzwx$. Denote

$$E = \{xy, yz, zw, wx\}.$$

Define G/π to be the graph obtained from $G - E$ as described in Section 4 above. Thus, G and G/π satisfy Theorem C.

Since $\delta(G) \geq 3$ and $d(x) \geq 4$, we have

(13) $$d_{G/\pi}(v_1) \geq 4 \quad \text{and} \quad \delta(G/\pi) \geq 3,$$

where v_1 is the vertex defined in Section 4. Let G_0 be the reduction of G/π. If $G = K_1$ then $G/\pi \in \mathcal{CL}$, and so (a) of Theorem C gives $G \in \mathcal{CL}$, contrary to the hypothesis of Lemma 4. Hence $G_0 \neq K_1$, and so by (b) of Theorem C,

(14) $$1 < |V(G_0)| \leq |V(G/\pi)| = |V(G)| - 2 = 10.$$

Case 1 Suppose that $\kappa'(G/\pi) < 2$. Then v_1v_2 is the only cut-edge of G/π, because G has no cut edge. Therefore, $G - E$ has two components, say G_1 and G_2, where $x, z \in V(G_1)$ and $y, w \in V(G_2)$.

Since the 4-cycle $xyzwx$ is an induced subgraph, $xz, wy \notin E(G)$. This, $\delta(G) \geq 3$, and the fact that G is simple imply that each G_i $(1 \leq i \leq 2)$ has a vertex of degree at least 3 that is not in $\{w, x, y, z\}$. Since G has order 12, since $\delta(G) \geq 3$, and since (10) precludes the presence of 3-cycles in G_i, this implies

$$5 \leq |V(G_i)| \leq 7, \qquad (1 \leq i \leq 2).$$

By $\delta(G) \geq 3$,

$$D_1(G_1) \cup D_1(G_2) \subseteq \{w, x, y, z\},$$

and these relations imply that each G_i, $1 \leq i \leq 2$, contains a nontrivial 2-edge-connected subgraph H_i, where H_i has at most two vertices of degree 2. Since $|V(H_i)| \leq 7$, Lemma 1 implies $H_i \in \mathcal{CL}$. Thus, H_i is a subgraph of G that contradicts (10).

Case 2 Suppose that $\kappa'(G/\pi) \geq 3$. Then $\kappa'(G_0) \geq \kappa'(G/\pi) \geq 3$. By this and (14), G_0 is nontrivial and satisfies the hypotheses of Lemma 2 and must therefore be the Petersen graph. This fact and (14) force $G_0 = G/\pi$, and so G/π is 3-regular, contrary to (13).

Case 3 Suppose that $\kappa'(G/\pi) = 2$. Since $\kappa'(G) \geq 3$, it follows that v_1v_2 is in every edge cut of size 2 in G/π. Denote $e_\pi = v_1v_2$. For the reduction G_0 of G/π, e_π lies in every edge cut of G_0 of size 2. By (b) of Theorem B,

(15) $$G_0 \text{ is simple.}$$

Subcase 3A Suppose that either $e_\pi \notin E(G_0)$ or $\kappa'(G_0) \geq 3$. In either case we must have $\kappa'(G_0) \geq 3$ and $1 < |V(G_0)| \leq 9$. This and (15) mean that G_0 is a counterexample to Lemma 2. Hence, Subcase 3A is impossible.

Subcase 3B Suppose that $e_\pi \in E(G_0)$ and $\kappa'(G_0) < 3$. Then

(16) $$\kappa'(G_0) = 2$$

and by a prior remark, e_π is in every edge cut of size 2 in G_0. If $\delta(G_0) \geq 3$, then by (14), (15), and Lemma 2, G_0 is the Petersen graph, contrary to (16). Hence,

(17) $$\delta(G_0) < 3.$$

Since e_π is in every edge cut of size 2 and by (16), (17) implies that G_0 has a unique vertex u (say) of degree 2, and u is incident with e_π. Denote $e_\pi = uv$ in $E(G_0)$.

3B(i). Suppose $|V(G_0)| \leq 8$. By (16) and by Lemma 3, $G_0 \in \mathcal{CL}$. Hence, $G/\pi \in \mathcal{CL}$ and by Theorem C, $G \in \mathcal{CL}$, contrary to the hypothesis of Lemma 5.

3B(ii). Suppose $|V(G_0)| \geq 9$. By (13), $\delta(G/\pi) \geq 3$, and so G/π has no vertex u of degree 2. Thus, G_0 is a proper contraction of G/π, and so by (14),

$$|V(G_0)| = 9, \qquad |V(G/\pi)| = 10.$$

Hence the contraction mapping $G/\pi \longrightarrow G_0$, being a reduction as well, identifies two vertices of $V(G/\pi)$ that are joined in G/π by multiple edges.

By the nature of the derivation of G/π from the simple graph G, any two vertices of G/π are joined by no more than two edges. Hence by the first part of (13), the contraction-mapping $G/\pi \longrightarrow G_0$ cannot involve an identification of v_1 with another vertex to form the vertex $u \in V(G_0)$, since u has degree 2. Instead, v_2 must be identified with a neighbor in G/π to form the vertex u in G_0, and so v_1 has degree at least 4 in G_0 as well as in G/π. Thus, $v = v_1$ in G_0. Let v' denote the other neighbor of u in G_0. Since e_π is in every edge-cut of size 2 in G_0, $\kappa'(G_0 - u) \geq 2$. By Lemma 3 (with $G_0 - u$ in place of G and with v' in place of u), $G_0 - u$ is collapsible of order 8. This contradicts the fact that G_0 is reduced. This contradiction concludes this subcase and it proves Lemma 4. □

Lemma 5 Let n be the smallest natural number such that there is a 2-edge-connected reduced graph G of order n and size $2n-4$, such that G is not $K_{2,n-2}$. Then $n \geq 14$ and G is 3-edge-connected.

Proof: Suppose that G is a smallest 2-edge-connected reduced graph with $|E(G)| = 2|V(G)|-4$, such that G is not $K_{2,n-2}$, where n denotes $|V(G)|$. Since G is reduced, $a(G) \leq 2$, by (b) of Theorem B. Hence, by (f) of Theorem B and by the definition of G,

$$F(G) = 2. \tag{18}$$

If $\delta(G) = 2$ then G has a vertex u of degree 2. If $\kappa'(G-u) < 2$ then since G is 2-edge-connected, $G-u$ has a cut edge e, say, and if G_1 and G_2 denote the components of $G-u-e$, then it follows from (18) that $F(G_1)+F(G_2) = 1$. By (d) of Theorem B and since G is reduced, $\{G_1, G_2\} = \{K_1, K_2\}$. Since G is 2-edge-connected, this forces $G = C_4$. Since this contradicts the hypothesis of the lemma, we may conclude that $\kappa'(G-u) \geq 2$. Hence, by the minimality of G, $G - u = K_{2,n-3}$. Since G is reduced, (e) of Theorem B implies that u is not in a subgraph that is a 2-cycle, a 3-cycle, or $K_{3,3}$ minus an edge, for these three graphs are collapsible. Since $G \neq K_{2,n-2}$, it follows that

$$\delta(G) \geq 3. \tag{19}$$

If $\kappa'(G) = 2$, then G has a cutset E of size 2, such that each component of $G - E$ is nontrivial, by (19). If $n < 14$ then the smallest component of $G - E$ satisfies the hypothesis of Lemma 1, and hence must be a nontrivial collapsible subgraph of G. This contradicts the hypothesis that G is reduced, and so $\kappa'(G) \neq 2$.

If $\kappa'(G) = 1$ then G has a cut edge e (say), and we denote by G_1 and G_2 the two components of $G - e$. By (18),

$$F(G_1) + F(G_2) = 1. \tag{20}$$

By (19), G_1 and G_2 are nontrivial, and by (20), one of them, say G_1, has $F(G_1) = 0$. By (d) of Theorem B, G_1 is a nontrivial collapsible subgraph of G, contrary to (e) of Theorem B, since $G = G'$. Hence, $\kappa'(G) \neq 1$, and so we must have

$$\kappa'(G) \geq 3.$$

Hence, if $n \leq 11$ then by Lemma 2, $G \in \mathcal{CL}$ or G is the Petersen graph. Either case violates the definition of G. If $n = 12$ then by Lemma 4, G is 3-regular, and so $|E(G)| = 18$, contrary to the definition of G. Hence, $n \geq 13$. Finally, therefore, we suppose

$$n = 13,$$

and we shall derive a contradiction.

We claim that G has a 4-cycle. Suppose not, and let x be a vertex of degree $d(x) = \Delta(G)$ in G. Since G is reduced, x is in no cycle of length less than 5. Thus, each path of length at most 2 with origin x has a different terminus. There are $d(x)$ such paths of length 1 and at least $2d(x)$ of length 2, since $\delta(G) \geq 3$ by (19). Hence,

$$12 = |V(G - x)| \geq d(x) + 2d(x) = 3d(x), \tag{21}$$

with equality only if each neighbor of x has degree 3. By (19), $\Delta(G) \geq 3$, and since G has odd order, G is not 3-regular. This and (21) imply that

$$d(x) = 4, \tag{22}$$

and since equality holds in (21), each vertex adjacent to x has degree 3 in G. Since x is arbitrary, no two vertices of degree 4 in G are adjacent.

By $|E(G)| = 2n - 4 = 22$, by (19), and by $\Delta(G) = 4$, G has 5 vertices of degree 4 and 8 vertices of degree 3. Define

$$H = G - (\{x\} \cup N(x)).$$

By (22) and since the four vertices of $N(x)$ have degree 3 in G, $V(H)$ consists of 8 vertices, of which 4 have degree 4 and 4 have degree 3 in G. Since G has exactly 8 paths of length 2 with origin x and since each of these paths has a distinct terminus in $V(H)$, each vertex of $V(H)$ is adjacent in G to exactly one vertex not in $V(H)$. Hence, $V(H)$ consists of 4 vertices of degree 3 in H, and 4 vertices of degree 2 in H. In H there are 12 incidences of edges at the 4 vertices of degree 3, and there are only 8 incidences at the 4 vertices of degree 2. Therefore, two vertices of degree 3 in H are adjacent. These are adjacent vertices of degree 4 in G, a contradiction. This contradiction proves the claim that G has a 4-cycle.

Let $xyzwx$ be an induced 4-cycle in G. Define the graph G/π as in Section 4, so that Theorem C holds. Define

$$E = \{wx, xy, yz, zw\},$$

and denote the edge v_1v_2 of G/π by e_π.

Case 1 Suppose that e_π is a cut-edge of G/π. Then $G - E$ is disconnected. Define G_1 and G_2 to be the two components of $G - E$, where $2 \leq |V(G_1)| \leq |V(G_2)|$. Since $n = 13$, $2 \leq |V(G_1)| \leq 6$, and by (19), G_1 has at most 2 vertices of degree less than 3. Therefore, G_1 has a nontrivial 2-edge-connected simple subgraph H_1, say, with at most two vertices of degree 2. By Lemma 1, $H_1 \in \mathcal{CL}$, and so G has a nontrivial collapsible subgraph. Since G is reduced, this violates (e) of Theorem B.

Case 2 Suppose that e_π is not a cut edge of G/π. We claim

$$a(G/\pi) \leq 2. \tag{23}$$

Suppose not. By Nash-Williams' arboricity formula [9], G/π has a subgraph H (say) with

$$|E(H)| \geq 2|V(H)| - 1. \tag{24}$$

Now since G is reduced, $a(G) \leq 2$, and so H contains one or both vertices of $\{v_1, v_2\}$.

Subcase 2A Suppose $V(H) \cap \{v_1, v_2\} = \{v_1\}$. Then

$$|V(G[E(H)])| = |V(H)| + 1, \tag{25}$$

and we combine (25) with (24) to get

$$\begin{aligned} |E(G[E(H)])| &= |E(H)| \geq 2|V(H)| - 1 \\ &= 2|V(G[E(H)])| - 3. \end{aligned}$$

Since $a(G) \leq 2$, it follows that $G[E(H)]$ is one edge short of having two edge-disjoint spanning trees, i.e., $F(G[E(H)]) = 1$. Since G is reduced, (d) of Theorem B implies $G[E(H)] = K_2$. By (25), this gives

$$|V(H)| = |V(G[E(H)])| - 1 = 1.$$

This and (24) imply $|E(H)| \geq 2|V(H)| - 1 \geq 1$, and since H has no loop, we have a contradiction.

Subcase 2B Suppose $v_1, v_2 \in V(H)$. Then

$$|V(G[E(H) \cup E])| = |V(H)| + 2. \tag{26}$$

By (24) and (26),

$$\begin{aligned} |E(G[E(H) \cup E])| &= |E(H)| + 3 \geq 2|V(H)| + 2 \\ &= 2|V(G[E(H) \cup E])| - 2. \end{aligned} \tag{27}$$

Since $a(G) \leq 2$, (27) implies that the subgraph $G[E(H) \cup E]$ has two edge-disjoint spanning trees, i.e., $F(G[E(H) \cup E]) = 0$. Such a subgraph is collapsible (by (d) of Theorem B), contrary to the fact that G is reduced. This contradiction concludes Subcase 2B and proves the claim (23).

By (23), (f) of Theorem B gives

$$|E(G/\pi)| + F(G/\pi) = 2|V(G/\pi)| - 2.$$

By Theorem C, since $n = 13$, and since $|E(G)| = 2n - 4$,

$$|E(G/\pi)| = |E(G)| - 3 = 19$$

and

$$|V(G/\pi)| = |V(G)| - 2 = n - 2 = 11,$$

and combining these, we get $F(G/\pi) = 1$. Since G/π is 2-edge-connected in Case 2, (d) of Theorem B gives $G/\pi \in \mathcal{CL}$. By (a) of Theorem C, $G \in \mathcal{CL}$, a contradiction, since G is reduced and nontrivial. Hence, $n \geq 14$, and Lemma 5 is proved. □

Catlin [5] conjectured that no smallest number n exists that satisfies the hypothesis of Lemma 5.

6. PROOF OF CAI'S CONJECTURE

Theorem 2 Let G be a simple 3-edge-connected graph of order n. If

$$(28) \qquad |E(G)| \geq \binom{n-9}{2} + 16,$$

then G is collapsible.

Proof: Let G satisfy the hypothesis of Theorem 2. If $G \in \mathcal{CL}$, then we are done. If not, then the reduction G' of G, has order at least 2 and is 3-edge-connected. By Lemma 2, either G' is the Petersen graph or G' has order $n \geq 12$.

But G also satisfies Theorem 1 with $p = 10$. By remarks of the prior paragraph, if conclusion (a) of Theorem 1 holds, then $G' = K_1$ and so $G \in \mathcal{CL}$. Conclusion (b) cannot hold, since the Petersen graph does not have size 16. If conclusion (c) holds, then G is a reduced graph of order $n \geq 12$, and either

$$|E(G)| \in \{19, 20\} \quad \text{and} \quad n = 12,$$

or

$$|E(G)| = 22 \quad \text{and} \quad n = 13.$$

By Lemma 4, if $n = 12$ then $|E(G)| = 18$, which is too small. By Lemma 5, if $n = 13$ and $|E(G)| = 22$ then $G = K_{2,11}$, contrary to the hypothesis that $\kappa'(G) \geq 3$. This exhausts the cases and proves Theorem 2. □

X. T. Cai [2] conjectured a weaker form of Theorem 2, in which "collapsible" is replaced by "supereulerian". It is easy to contruct graphs to show that (28) is best-possible, both in Theorem 2 and in Cai's conjecture. Let G be the simple graph obtained from a Petersen graph and the complete graph K_{n-9} by identifying one vertex from each graph. Then G has order $n = (n-9) + 10 - 1$, and if $n = 10$ or if $n \geq 13$ then $\kappa'(G) \geq 3$. Also,

$$|E(G)| = \binom{n-9}{2} + 15,$$

and since the reduction of G is the Petersen graph, G is not collapsible and (by (a) of Theorem B) G is not supereulerian. Hence, (28) is sharp.

7. CONCLUDING REMARKS

Theorem D (Cai [2]) Let G be a 2-edge-connected simple graph of order n. If

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{29}$$

then exactly one of the following holds:

(i) $G \in \mathcal{SL}$;

(ii) Equality holds in (29) and G has a complete subgraph H of order $n-4$ such that $G/H = K_{2,3}$;

(iii) G is either $K_{2,5}$ or the cube minus a vertex.

Proof: Let G be a 2-edge-connected graph of order n satisfying (29), and let G' be the reduction of G. Then G satisfies the hypothesis of Theorem 1 with $p = 5$. If conclusion (a) of Theorem 1 holds, then G' is a 2-edge-connected reduced graph of order less than 5, and so $G' = K_1$. Hence, by (a) of Theorem B, $G \in \mathcal{SL}$. If (b) of Theorem 1 holds, then equality holds in (29) and G has a complete subgraph H of order $n-4$ such that G' is G/H, a graph of order 5 and size 6. By Lemma 5, $G/H = K_{2,3}$. If (c) holds, then G is a reduced graph such that either

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{6,7\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = 8.$$

By Lemma 5, if $|E(G)| = 2n-4$ for $n \in \{6,7,8\}$ then $G = K_{2,n-2}$, and so either $G \in \mathcal{SL}$ or $G = K_{2,5}$. If $|E(G)| = 2n-5$ and $n = 6$, then since $G = G'$ is 2-edge-connected and satisfies (b) of Theorem B, either G is a cube minus two adjacent vertices (hence in $\mathcal{SL}$) or G is contractible to $K_{2,3}$. If $|E(G)| = 2n-5$ and $n = 7$, then since $G = G'$ is 2-edge-connected and satisfies (b) of Theorem B, G is a cube minus a vertex. □

There are four contraction-minimal nonsupereulerian graphs of order at most 7, namely K_2, $K_{2,3}$, $K_{2,5}$ and $Q_3 - v$ (the cube minus a vertex). A consequence of this fact and Theorem 1 (with $p = 7$) is this:

Theorem 3 Let G be a connected simple graph of order $n \geq 10$. If

$$|E(G)| \geq \binom{n-6}{2} + 10, \tag{30}$$

then exactly one of the following holds:

(i) $G \in \mathcal{SL}$;

(ii) G is contractible to K_2 or $K_{2,3}$;

(iii) Equality holds in (30), G has a complete subgraph H of order $n-6$, and $G/H = K_{2,5}$. □

Conclusion (c) of Theorem 1 is precluded by the hypothesis $n \geq 10$ and because the only 2-edge-connected reduced graph of order $n = 10$ and size 16 is $K_{2,8}$ (by Lemma 5), which is supereulerian. There are several graphs of orders 8 and 9 that violate (30) and conclusions (i), (ii), and (iii). To see that (30) is best-possible, let G be a simple graph containing the complete subgraph $H = K_{n-6}$, $n \geq 10$, such that $G/H = Q_3 - v$. Then (30) barely fails and conclusions (i), (ii), and (iii) fail.

Veldman [10] uses lower bounds on $|E(G)|$ similar to those in this paper, in order to show that a given graph G has a cycle containing at least one end of each edge of G.

8. REFERENCES

1. J. A. BONDY and U. S. R. MURTY, Graph Theory with Applications, American Elsevier, New York (1976).

2. X. T. CAI, Connected eulerian spanning subgraphs, Chinese Quart. J. of Math., to appear.

3. P. A. CATLIN, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29-45.

4. P. A. CATLIN, Supereulerian graphs, collapsible graphs, and four-cycles, Congressus Numerantium 58 (1988) 233-246.

5. P. A. CATLIN, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465-483.

6. P. A. CATLIN and H.-J. LAI, Spanning trails joining two given edges, Proc. 6th Internat. Conference on Graph Theory, John Wiley & Sons, 127-138.

7. Z.-H. CHEN, Supereulerian graphs and the Petersen graph, submitted.

8. Z.-H. CHEN and H.-J. LAI, Reductions of small graphs.

9. C. ST. J. A. NASH-WILLIAMS, Decomposition of finite graphs into forests, J. London Math. Soc. 36 (1964) 12.

10. H. VELDMAN, Existence of dominating cycles and paths, Discrete Math. 43 (1983) 281-296.