Embedded Graphs, Facial Colorings, and Double Cycle Covers

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We use the notation of Bondy and Murty [4], except that graphs have no loops. Let H be a subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H and deleting any resulting loops. Denote

$$O(G) = \{\text{odd-degree vertices of } G\}.$$

The graph G is called an even graph if $O(G) = \emptyset$, and an even graph is called supereulerian if it also connected. Denote

$$S\mathcal{L} = \{\text{supereulerian graphs}\}.$$

Let G be a graph embedded on a surface. A coloring of the faces is called packed if at each vertex of degree at least 3, the incident faces have been assigned at least three different colors, and if each edge is incident with faces of two different colors. If G is 3-regular with no cut edge, then any proper facial coloring is packed. We shall generalize a criterion of Archdeacon in order to give a reduction method to determine whether a graph has a packed facial 3-coloring or 4-coloring on some surface.

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The subgraph H of a graph G is said to <u>evenly span</u> G if

(i) each vertex of G is of even degree in H;

(ii) each vertex of degree at least 3 in G has nonzero degree in H; and

(iii) each component of H contains evenly many vertices of $O(G)$.

We say that G is <u>evenly spanned</u> if some subgraph H evenly spans G. Denote

$$\mathcal{ES} = \{\text{evenly spanned graphs}\}.$$

Archdeacon [2] noted that $\mathcal{SL} \subset \mathcal{ES}$, and he proved for graphs with minimum degree at least 3 that membership in $\mathcal{ES}$ is equivalent to the existence of a packed 3-coloring of the faces of some embedding on some surface. The smallest 2-edge-connected graph not evenly spanned is the Petersen graph.

Let G be a graph with no cut edge. A <u>3-splitting</u> of G is any 3-regular graph H that can be converted to G by a sequence of edge contractions and edge-subdivisions.

The following result was first proved by Tutte [15] for 3-regular graphs, and Archdeacon [2] obtained the present version:

<u>Theorem 1</u> [2] Let G be a graph with $\delta(G) \geq 3$. The following are equivalent:

(a) $G \in \mathcal{ES}$;

(b) G has a 3-splitting H with $\chi'(H) = 3$;

(c) G embeds on some orientable surface with a packed 4-coloring of the faces;

(d) G embeds on some surface with a packed 4-coloring of the faces;

(e) G embeds on some surface with a packed 3-coloring of the faces. □

Define $\mathcal{S}_3$ to be the family of graphs G such that $G \in \mathcal{S}_3$ whenever there is a par-

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<u>Theorem 2</u> Let G b

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(f) $G \in \mathcal{S}_3$;

(g) G has a 3-color

(h) G has a 4-color

<u>Proof</u>: Any graph

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tition $E(G) = E_1 \cup E_2 \cup E_3$ such that $O(G[E_k]) = O(G)$ for all $k \in \{1,2,3\}$. In [9] we noted that $\mathcal{SL} \subset \mathcal{S}_3$ properly. A double cycle cover of G is a collection of cycles of G (multiplicities allowed) such that each edge of G is in exactly two cycles in the collection. A double cycle cover of G is called k-colorable, $k \geq 2$, if the collection of cycles can be partitioned into k subcollections, where the cycles in each subcollection are edge-disjoint (and hence induce an even subgraph of G).

Theorem 2 Let G be a graph. Then conditions (a) through (e) of Theorem 1 are equivalent to each other, and they are equivalent to each of these:

(f) $G \in \mathcal{S}_3$;

(g) G has a 3-colorable double cycle cover;

(h) G has a 4-colorable double cycle cover.

Proof: Any graph with a vertex of degree 1 satisfies none of the conditions (a) through (h). Suppose that G is a smallest counterexample with $\delta(G) = 2$. By the minimality of G, no component and no endblock of G is a cycle. Therefore, there is a graph G' with $\delta(G') \geq 3$ such that G can be obtained from G' by subdivisions of edges. For each of the conditions (a) through (h), it is easy to check that G satisfies that condition if and only if G' does. Thus, the first part of Theorem 2 follows by applying Theorem 1 to G'.

The equivalence of (f) and (g) is easy, since $G - E_k$ is an even graph if and only if $O(G[E_k]) = O(G)$, where $k \in \{1,2,3\}$. The equivalence of (g) and (h) was proved by Bermond, Jackson, and Jaeger [3].

To prove that (f) is equivalent to the other conditions, we shall prove (f) $\Leftrightarrow$ (a) by proving $S_3 = \mathcal{ES}$. First, suppose that $G \in \mathcal{ES}$. Then G has a subgraph H that evenly spans G, and hence that satisfies conditions (i), (ii), and (iii) of the definition. Define $E_3 = E(G) - E(H)$. Then $O(G[E_3]) = O(G)$, by (i). Since H is an even graph, it is an easy consequence of (iii) that there is a partition $E(H) = E_1 \cup E_2$ such that $O(G[E_k]) = O(G)$, $k \in \{1, 2\}$. Therefore, $G \in S_3$. This proves $\mathcal{ES} \subseteq S_3$.

Before proving $S_3 \subseteq \mathcal{ES}$, we first present a lemma and some notation. Let H be a graph, let $v \in V(G)$, and let e and e' be distinct edges of H that are incident with v. Denote by $H(v, e, e')$ the graph with edge set $E(H)$ and with vertex set $V(H) \cup \{v'\}$, where $v' \notin V(H)$, where e and e' are incident in $H(v, e, e')$ with v' instead of v, and where all other incidences in $H(v, e, e')$ are the same as in H. This lemma is an easy consequence of the definitions:

<u>Lemma</u> For any graph H, for any $v \in V(H)$, and for any pair $\{e, e'\}$ of distinct edges of H incident with v, H is an even graph if and only if $H(v, e, e')$ is an even graph. □

Let G be a graph, and let $E(G) = E_1 \cup E_2 \cup E_3$ and $E(G) = E'_1 \cup E'_2 \cup E'_3$ be two partitions of $E(G)$. For any vertex $v \in V(G)$ and for each $k \in \{1, 2, 3\}$, denote by $d_k(v)$ (respectively, $d'_k(v)$) the number of incidences of v with edges of E_k (respectively, E'_k).

Suppose that $G \in S_3$. Then there is a partition

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$$(2) \qquad d_k(v) \equiv d(v) \pmod 2$$

for all $v \in V(G)$ and $k \in \{1,2,3\}$. Define the set $T = T(E_3)$ by

$$T = \{x \in V(G) \mid d_3(x) = d(x) \geq 3\},$$

and suppose that the partition $E_1 \cup E_2 \cup E_3$ satisfying (1) and (2) has been chosen to minimize $|T|$. We claim that T is empty.

By way of contradiction, suppose that $|T| > 0$. Then G has a vertex $x \in T$, where $d_3(x) = d(x) \geq 3$. Define the set

$$S = \{v \in V(G) \mid d_1(v) = 2 \text{ and } d_3(v) = d(v) - 2 > 0\}.$$

If $v \in S$ then v is incident with edges e_1 and e_3 (say), where $e_1 \in E_1$ and $e_3 \in E_3$. By (1) and (2), $G[E_1 \cup E_3]$ is an even subgraph of G, and so the lemma implies that $G[E_1 \cup E_3](v, e_1, e_3)$ is an even subgraph of $G(v, e_1, e_3)$, a graph of order $|V(G)| + 1$. Apply the lemma in this way at each vertex $v \in S$, to convert G into a graph G_0, say, of order $|V(G)| + |S|$. Thus, $G_0[E_1 \cup E_3]$ is an even subgraph of G_0, and each $v \in V(G_0) - V(G)$ is incident in G_0 with exactly two edges, one in E_1 and one in E_3.

Since $G_0[E_1 \cup E_3]$ is an even graph it has no cut edge, and so the vertex x is in a cycle C, say, in $G_0[E_1 \cup E_3]$. We may also regard C as a closed trail in $G[E_1 \cup E_3]$ on the same edge set. Define

$$E_1' = E_1 \triangle E(C); \quad E_2' = E_2; \text{ and } \quad E_3' = E_3 \triangle E(C),$$

where $E_k \triangle E(C)$ denotes the symmetric difference of E_k and $E(C)$. Then $E_1' \cup E_2' \cup E_3'$ is a partition of $E(G)$ satisfying $d_k'(v) \equiv d(v) \pmod 2$ for all $v \in V(G)$ and $k \in \{1,2,3\}$.

Denote $T' = T(E_3')$. We claim $T' \subset T$ properly. If $v \in V(G) - V(C)$ then $d_3'(v) = d_3(v)$ and $v \in T \not\Leftrightarrow v \in T'$. If $v \in V(C) \cap S$, then by the construction of the even graph $G_0[E_1 \cup E_3]$ containing C, v is incident with at least one edge in $E_1 \triangle E(C)$. Hence $v \notin T'$. If $v \in (V(C) \cap T) - S$ then $d(v) - 2 = d_3(v) - 2 = d_3'(v)$ and $d_1'(v) = 2$, and so $v \notin T'$. If $v \in V(C) - (S \cup T)$ then either $d_3'(v) \neq d(v)$ or $d(v) < 3$, and so $v \notin T'$. Hence $T' \subset T$, and since $x \in T - T'$, this containment is proper, contrary to the minimality of $|T|$. Thus, $T = \emptyset$, as claimed.

Define $H = G[E_1 \cup E_2]$. We claim that H evenly spans G. By (1) and (2), H satisfies (i) of the definition. Since T is empty, H satisfies (ii). If a component H_0 of H contains an odd number of vertices of $O(G)$, then by (2), $O(H_0[E_1])$ is that odd set of vertices in $O(G)$, an impossibility since $|O(H[E_1])|$ is even. Thus, (iii) holds and the claim is proved. Hence $G \in \mathcal{ES}$ and so $\mathcal{S}_3 \subseteq \mathcal{ES}$. □

Corollary [2] If G satisfies any of the conditions (a) through (e), then G has a double cycle cover. □

Define $\mathcal{S}_3^O$ to be the family of graphs H such that for any supergraph G of H, this equivalence holds:

$$G \in \mathcal{S}_3 \iff G/H \in \mathcal{S}_3. \tag{3}$$

Let $\mathcal{C}_3$ be the family of graphs G such that for any even subsets $X, Y \subseteq V(G)$ there are disjoint subsets $E_X, E_Y \subseteq E(G)$ such that $O(G[E_X]) = X$ and $O(G[E_Y]) = Y$. It was proved [7, Corollary 13A] that $\mathcal{C}_3 \cup \{C_4\} \subseteq \mathcal{S}_3^O$, and so $\mathcal{S}_3^O$ contains all cycles of length at most 4. Also, $\mathcal{C}_3$ contains every graph that has two edge-disjoint spanning trees. By repeated applications of (3), the question of membership in $\mathcal{S}_3$ can be reduced to

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the consideration of graphs with no nontrivial subgraphs in $\mathcal{C}_3$. An analogous reduction method for determining membership in $\mathcal{SL}$ was introduced in [5]. Both of these reduction methods are special cases of a more general method discussed in [6] and [8].

A lifting theorem of Fleischner [10] (see [11], p. 4) can also be used to investigate membership in $\mathcal{S}_3$. It can be applied either before or after (3) has been used repeatedly. Suppose that v is a vertex of degree at least 4 in G. By Fleischner's Theorem [10], for some pair of distinct edges e and e' incident with v, $G \in \mathcal{S}_3$ if and only if $G(v, e, e') \in \mathcal{S}_3$.

Szekeres [14] and Seymour [13] conjectured that any graph with no cut edge has a double cycle cover. For graphs with no cut edge and having no subgraph that is a subdivision of the Petersen graph, Matthews [12] (generalizing a conjecture of Tutte [16] on 3-regular graphs) conjectured that they are in $\mathcal{S}_3$, and Alspach and Zhang [1] showed that they have a double cycle cover.

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