Double cycle covers and the Petersen graph, II

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Abstract

Any graph with no cut edge and with at most 13 edge cuts of size 3 either has a double cycle cover formed by 3 subgraphs whose components are eulerian, or it is contractible to the Petersen graph.

INTRODUCTION

For a graph G (with multiple edges allowed, but loops eschewed) we denote

$$O(G) = \{\text{odd degree vertices of } G\}.$$

A graph G is eulerian if $O(G) = \emptyset$ and G is connected. A graph G is even if $O(G) = \emptyset$. Define $\mathcal{S}_3$ to be the family of graphs G for which there is a partition $E(G) = E_1 \cup E_2 \cup E_3$ satisfying

(1) $$O(G[E_i]) = O(G), \quad 1 \leq i \leq 3.$$

The smallest 2-edge-connected graph not in $\mathcal{S}_3$ is the Petersen graph, called P.

A graph in $\mathcal{S}_3$ has a double cycle cover whose cycles can be partitioned into three sets, each inducing an even subgraph of G. Each of the three classes in this partition induces a subgraph of G of the form $G - E_i$, $1 \leq i \leq 3$, where E_i is the set of the definition of $\mathcal{S}_3$. Conversely, a graph not in $\mathcal{S}_3$ has no such double cycle cover.

Let G be a graph. Assign a direction to each edge of G. For $k \geq 2$, a nowhere zero k-flow is an assignment of nonzero members of $\mathbf{Z}_k$ to $E(G)$ (i.e., an assignment of nonzero weights from the ring $\mathbf{Z}_k$) such that at each vertex of this directed graph, the sum of the weights of the incoming edges minus the sum of the weights on the outgoing edges is zero. Thus, G has a nowhere-zero 2-flow if and only if each component of G is eulerian, and one can check that G has a nowhere-zero 4-flow if and only if $G \in \mathcal{S}_3$. (It is very easy to show that $G \in \mathcal{S}_3$ if and only if G has a "D_3-flow", in the terminology of [10]. What is called a nowhere-zero 4-flow here is called a nowhere-zero $\mathbf{Z}_4$-flow in [10]. The equivalence of "$G \in \mathcal{S}_3$" and "G has a 4-flow" in this paper then follows from the equivalence $(i) \iff (ii)$ of Theorem 5.5 of [10].) The assignment of directions to the edges can be arbitrary, for if a direction assigned to an edge is reversed, then we can always change the sign of the weight that we assign

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to that edge. If a graph G has a nowhere-zero k-flow, then for any larger integer h, G has a nowhere-zero h-flow. Seymour [13] showed that any graph with no cut edge has a nowhere-zero 6-flow. For a survey of results on nowhere-zero k-flows, see [10].

In this paper our primary interest concerns $\mathcal{S}_3$, the family of graphs having 4-flows. The following conjecture was made by Tutte [14,15] and by Matthews [12].

Conjecture 1 [14,15], [12] Let G be a graph with no cut edge. If $G \notin \mathcal{S}_3$ then some subgraph of G is contractible to P.

A bond of G is any minimal set E of edges of G such that $G - E$ has more components than G. The size of a bond is the number of edges in it.

Theorem 1 Let G be a 2-edge-connected graph, and let E be a nontrivial bond of size at most 3 in G. Denote the components of $G - E$ by G_1 and G_2. Then

$$G \in \mathcal{S}_3 \iff \text{Both } G/G_1 \in \mathcal{S}_3 \text{ and } G/G_2 \in \mathcal{S}_3. \quad \Box$$

Theorem 1 is not new (see, e.g., [9] for a partial proof), but for completeness, we outline a proof here. The cases $|E| = 2$ and $|E| = 3$ are similar, and so we shall just consider the case $|E| = 3$. A parity argument shows that $|O(G) \cap V(G_i)|$ is odd for $i \in \{1, 2\}$, when $|E|$ is odd. If $G/G_1 \in \mathcal{S}_3$ and $G/G_2 \in \mathcal{S}_3$, then there are partitions $E_1^i \cup E_2^i \cup E_3^i$ of $E(G/G_i)$, $i \in \{1, 2\}$, satisfying (1). These partitions can be "pieced together" to produce a partition $E(G) = E_1 \cup E_2 \cup E_3$, such that (1) holds. Hence, $G \in \mathcal{S}_3$. Conversely, suppose $G \in \mathcal{S}_3$. Then there is a partition $E(G) = E_1 \cup E_2 \cup E_3$ satisfying (1). It induces corresponding partitions of $E(G/G_1)$ and $E(G/G_2)$, to show that $G/G_1, G/G_2 \in \mathcal{S}_3$.

A bond is called trivial if all of its edges are incident with a single vertex. We call a graph essentially k-edge-connected if all bonds of size less than k are trivial. For example, P is essentially 4-edge-connected. Theorem 1 reduces the problem of determining which graphs are in $\mathcal{S}_3$ to the case of essentially 4-edge-connected graphs. Also, the contractions of Theorem 1 will not increase the number of bonds of size 3. Therefore, we lose no generality in the following theorem in restricting our attention to graphs that are essentially 4-edge-connected.

Jaeger [8] proved that any graph with no cut edge and with no bond of size 3 is in $\mathcal{S}_3$. Tutte (see [1], unsolved problem) conjectured that any such graph also has a 3-flow. We shall improve Jaeger's result as follows:

Theorem 2 Let G be an essentially 4-edge-connected graph with no cut edge. If G has at most 13 bonds of size 3, then either $G \in \mathcal{S}_3$ or $G = P$.

The proof of Theorem 2 appears in a subsequent section.

Corollary 2A If a graph G with no cut edge has at most 13 bonds of size 3, then exactly one of these holds:

(a) $G \in \mathcal{S}_3$;

(b) G is contractible to the Petersen graph.

Proof: Combine Theorems 1 and 2 with $P \notin \mathcal{S}_3$. □

Any graph that satisfies the hypothesis of either Theorem 2 or Corollary 2A also satisfies Conjecture 1.

THE REDUCTION METHOD

Call a graph supereulerian if it has a spanning eulerian subgraph. Denote

$$\mathcal{SL} = \{\text{supereulerian graphs}\}.$$

Also, call G collapsible if for every even set $X \subseteq V(G)$, G has a spanning connected subgraph H_X such that $O(H_X) = X$. Denote

$$\mathcal{CL} = \{\text{collapsible graphs}\}.$$

For a graph G with a subgraph H, denote by G/H the graph obtained from G by contracting all edges in H and by deleting any resulting loops. A reduction technique for determining membership in $\mathcal{S}_3$ is based on these two results:

Theorem 3 [3] Let H be a subgraph of G. If $H \in \mathcal{CL}$ or if H is a 4-cycle, then

$$G \in \mathcal{S}_3 \iff G/H \in \mathcal{S}_3. \ \square$$

Theorem 4 [2] If H is at most one edge short of having two edge-disjoint spanning trees, then exactly one of these holds:

(a) $H \in \mathcal{CL}$;

(b) H has a cut edge. □

For a graph G, let $F(G)$ be the smallest number of edges that must be added to G, to create a graph with two edge-disjoint spanning trees. The edge-arboricity of a graph G is the minimum number of edge-disjoint forests whose union contains G. In [2], we showed that a graph with no nontrivial collapsible subgraph has edge-arboricity at most 2. An easy consequence is the following result:

Theorem 5 Let G be a graph of order n. If G has no nontrivial collapsible subgraph, then

$$|E(G)| + F(G) = 2n - 2. \ \square$$

We shall use the following result, which has not yet appeared:

Theorem 6 (Catlin and Lai [4]) $\mathcal{SL} \subset \mathcal{S}_3$.

Proof: Let $G \in \mathcal{SL}$. Then G has a spanning eulerian subgraph H_1. Denote

$$E(G) - E(H_1) = \{e_1, e_2, ..., e_t\},$$

and let T be a spanning tree of H_1 (and hence of G). Denote by C_i the unique cycle of $T + e_i$ $(1 \leq i \leq t)$, and consider the symmetric difference

$$H_2 = G[E(C_1) \triangle E(C_2) \triangle .... \triangle E(C_t)].$$

Clearly, H_2 is an even subgraph of G containing $E(G) - E(H)$. Define

$$E_1 = E(H_1) - E(H_2), \quad E_2 = E(H_2) - E(H_1), \quad E_3 = E(H_1) \cap E(H_2).$$

Then $O(G[E_1]) = O(G)$ $(1 \leq i \leq 3)$, and so $G \in \mathcal{S}_3$, since $E(G) = E_1 \cup E_2 \cup E_3$. The cube minus a vertex is in $\mathcal{S}_3 - \mathcal{SL}$, and so containment is proper. □

The proof above of Theorem 6 is due to Lai. A different proof of Theorem 6 involves lifting edges (a concept defined below) to convert a spanning eulerian subgraph H_1 of G into a hamilton cycle of a related graph, but the details are omitted here. It is straightforward to show that the resulting hamiltonian graph is in $\mathcal{S}_3$ and that as a result, $G \in \mathcal{S}_3$. E. Palmer noted that an earlier "proof" of ours of Theorem 6 was incorrect.

MADER'S REDUCTION AND SNARKS

Let G be a graph. For any distinct vertices $v, w \in V(G)$, define $\kappa'(v,w)$ to be the minimum number of edges in $E(G)$ whose removal separates v and w.

For any graph G with edges xy and yz incident with a vertex y, the graph obtained from $G - \{xy, yz\}$ by adding a new edge xz is called the graph obtained from G by lifting $\{xy, yz\}$. The pair $\{xy, yz\}$ is said to be lifted. If also that vertex y has degree 2, then we say that y is dissolved when the pair $\{xy, yz\}$ is lifted and y is deleted.

Theorem 7 (Mader [11]) Suppose that $y \in V(G)$ is not a cutvertex of G. If $d(y) \geq 4$, then some pair of edges incident with y can be lifted so that in the resulting graph G_0, any pair of distinct vertices $v, w \in V(G) - y$ satisfy

$$\kappa'_{G_0}(v,w) = \kappa'_G(v,w). \tag{2}$$

If $d(y) = 2$, then (2) holds when y is dissolved. □

A snark is any 3-regular essentially 4-edge-connected graph of girth at least 5 that is not in $\mathcal{S}_3$. Since 3-regular graphs have even order, snarks have even order.

Theorem 8 [7] The only snark of order less than 18 is P. □

Let G be a graph with a cutvertex y. Let G_1 and G_2 be nontrivial subgraphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{y\}$. A graph that is the disjoint union of copies of G_1 and G_2 will be said to be obtained from G by cleaving G at y.

The following result is a straightforward consequence of Theorem 7 and of the definition of $\mathcal{S}_3$. By repeated applications, it reduces the question of membership in $\mathcal{S}_3$ to the special case of 3-regular graphs. (This reduction is basically due to Fleischner [6].)

Theorem 9 Let G be a graph with no cut edge and let y be a vertex whose degree is either 2 or at least 4. If $d(y) \geq 4$ and if y is not a cutvertex of G, then define G_0 as in Theorem 7. If $d(y) = 2$, then let G_0 be the graph obtained when y_0 is dissolved. If y is a cut-vertex, then define G_0 to be a graph obtained from G by cleaving G at y. In either case, each of these holds:
(a) G_0 has no cut edge;
(b) $G_0 \in \mathcal{S}_3 \implies G \in \mathcal{S}_3$;
(c) If each block of G is 3-edge-connected, then each block of G_0 is 3-edge-connected;
(d) If each block of G is essentially 4-edge-connected, then each block of G_0 is essentially 4-edge-connected. □

PROOF OF THEOREM 2

Suppose that G is a smallest counterexample to Theorem 2. Thus, G is 2-edge-connected, G is essentially 4-edge-connected, G is not P, $G \notin \mathcal{S}_3$, and G has at most 13 bonds of size 3.

Lemma 10 No nontrivial subgraph of G is in $\mathcal{CL} \cup \{C_4\}$.

Proof: By way of contradiction, suppose that G has a nontrivial subgraph H in $\mathcal{CL} \cup \{C_4\}$. By Theorem 9, since G is 2-edge-connected, so is G/H; and since G is essentially 4-edge-connected, so is G/H. Since G is essentially 4-edge-connected, $G/H \neq P$. By $G \notin \mathcal{S}_3$ and by Theorem 3, $G/H \notin \mathcal{S}_3$. Also, G/H cannot have more bonds of size 3 than G. Hence, G/H is a smaller counterexample to Theorem 2, and we have a contradiction. □

Lemma 11 The girth of G is at least 5.
Proof: Use Lemma 10 and the fact that $\mathcal{CL}$ contains the 2-cycle and the 3-cycle. □

Lemma 12 The graph G is 3-edge-connected.

Proof: By way of contradiction, suppose that G has a bond E of size 2. Denote the components of $G - E$ by G_1 and G_2. Since G is essentially 4-edge-connected, E is a trivial bond, and so we lose no generality in assuming that G_1 is just a single vertex v of degree 2 in G. Contract an edge incident with v to get a smaller counterexample, a contradiction. □

Lemma 13 All vertices of G have odd degree.

Proof: If $y \in V(G)$ has even degree, then define G_0 as in Theorem 9. It can be seen that some component of G_0 is a counterexample with fewer edges than G, contrary to the minimality of G. □

Let n_3 be the number of vertices of degree 3 in G, and let n_5 be the number of vertices of degree at least 5. By Lemmas 12 and 13,

$$(3) \qquad n_3 + n_5 = n.$$

Counting edge-vertex incidences of G in two ways, we get from (3) that

$$(4) \qquad 2|E(G)| \geq 3n_3 + 5n_5 = 4n + n_5 - n_3,$$

with equality only if $\Delta(G) \leq 5$. By Lemma 10, by Theorem 5, and by (4),

$$(5) \qquad F(G) = 2n - 2 - |E(G)| \leq \frac{1}{2}(n_3 - n_5) - 2,$$

with equality only if $\Delta(G) \leq 5$.

Case 1 Suppose that $F(G) \leq 1$. Since G has no cut edge, Theorem 4 gives us $G \in \mathcal{CL}$. By definitions, $\mathcal{CL} \subseteq \mathcal{SL}$, and by Theorem 6, $\mathcal{SL} \subseteq \mathcal{S}_3$. Hence, $G \in \mathcal{S}_3$, contrary to the supposition that G is a counterexample.

Case 2 Suppose that $n \leq 17$. If $n_5 = 0$, then G is 3-regular with girth at least 5 (by Lemma 11) and essential edge-connectivity at least 4. Hence, G is a snark of order less than 18. By Theorem 8, $G = P$, a contradiction. Hence, $n_5 > 0$.

Let y be a vertex of degree at least 5. Apply Theorem 9 to obtain G_0. Since $G \notin \mathcal{S}_3$, (b) of Theorem 9 gives us $G_0 \notin \mathcal{S}_3$. By (c) and (d), by Lemma 12, and since G is essentially 4-edge-connected, G_0 is 3-edge-connected and essentially 4-edge-connected. We can repeat this procedure (of lifting pairs of edges incident with vertices of degree at least 5 and applying Theorem 9) until we finally obtain a graph, say G_1, with no vertex of degree at least 5. Then G_1 will also be 3-edge-connected and essentially 4-edge-connected, and it follows from Lemma 13 that G_1 is 3-regular. Also, $G_1 \notin \mathcal{S}_3$. Therefore, G_1 is a snark. Now the order of G_1 is the same as the order of G, and by the hypothesis of this case, its order is less than 17. Hence, by Theorem 8, $G_1 = P$, and so G has order $|V(P)| = 10$. Since $n_5 > 0$ and since G is a graph of order 10 that can be converted to P by lifting pairs of edges, it is routine to show that the girth of G is less than 5. This contradicts Lemma 11, and so Case 2 fails.

Case 3 Suppose that Cases 1 and 2 do not apply. Since Case 1 does not apply, we have $F(G) \geq 2$, and so (5) gives

$$n_3 - n_5 \geq 8,$$

with equality only if $\Delta(G) \leq 5$. By (3) and since Case 2 does not apply,

$$n_3 + n_5 = n \geq 18.$$

By the hypothesis of Theorem 2, $n_3 \leq 13$. When combined, these relations force

$$n_3 = 13; \quad n_5 = 5; \quad n = 18; \quad \Delta(G) = 5.$$

Hence, all 18 vertices have degree either 3 or 5. Denote by S_5 the set of 5 vertices of degree 5, and denote by S_3 the set of 13 vertices of degree 3 in G.

Subcase 3A Suppose that S_5 is an independent set in G. There are 25 edges incident with S_5, and they join the 5 vertices of S_5 with the 13 vertices of S_3. It is routine to check that some 4 of these 25 edges induce a 4-cycle in G (or that two of these edges are parallel), contrary to Lemma 11.

Subcase 3B Suppose that $G[S_5]$ contains a vertex of degree at least 2, say w. Then $|N(w)| = 5$, and since G has girth at least 5 (by Lemma 11), the number of vertices at distance 2 from w is

$$4|S_5 \cap N(w)| + 2|S_3 \cap N(w)| \geq 4(2) + 2(3) = 14.$$

Hence, G has order at least $1 + |N(w)| + 14$, contrary to $n = 18$.

Subcase 3C Suppose that Subcases 3A and 3B do not apply. Then S_5 contains a pair $\{u, v\}$ of adjacent vertices of degree 5 in G, and their neighbors are in S_3. By Lemma 11,

$$|(N(u) \cup N(v)) - \{u, v\}| = 8.$$

Let R be the set of vertices at distance 2 from $\{u, v\}$. Thus,

$$18 = n \geq |\{u, v\}| + |(N(u) \cup N(v)) - \{u, v\}| + |R| = 10 + |R|,$$

and so $|R| \leq 8$, with equality only if there is no vertex at distance 3 from $\{u, v\}$. There are 8 edges of G with one end in $N(u) - v$ and the other end in R, and by Lemma 11, no two of them are incident with a common vertex of R. Hence, $|R| = 8$, each vertex of R is adjacent to exactly one vertex of $N(u) - v$, and there is no vertex at distance 3 from $\{u, v\}$. Likewise, each vertex of R is adjacent to exactly one vertex of $N(v) - u$. Thus, $G[R]$ has $|R \cap S_5| = 3$ vertices of degree 3 and $|R \cap S_3| = 5$ vertices of degree 1. Since $G[R]$ thus has degree sequence $(1, 1, 1, 1, 1, 3, 3, 3)$ and since $G[R]$ has no cycle of length at most 3 (Lemma 11), it can be checked easily that $G[R]$ must have a vertex $w \in R \cap S_5$ adjacent to two other vertices of $R \cap S_5$. Thus, Subcase 3B holds for w, a contradiction. This concludes Case 3, and the proof of Theorem 2. □

OTHER REMARKS

A weaker version of Corollary 2A, with "10" in place of "13", can be shown to be equivalent to Theorem 14 of [3].

Conjecture 2 Theorem 2 remains true if "13" is replaced by "17".

Snarks of order 18 (see, e.g. [5]) show that "17" is best possible in Conjecture 2. One can imitate the proof of Theorem 2, with "at most 17 bonds of size 3" in place of "at most 13 bonds of size 3", to show that if Conjecture 2 were false, then the smallest counterexample would have order between 18 and 26.

Conjecture 3 Let G be a 3-edge-connected graph. If G has at most 17 bonds of size 3, then either $G \in \mathcal{SL}$ or G is contractible to P.

The following result has been obtained by Catlin and Lai [4]:

Theorem 14 Let G be a 2-edge-connected graph. If $F(G) \leq 2$ then exactly one of the following holds:
(a) $G \in \mathcal{SL}$;
(b) G is contractible to $K_{2,t}$, where $t \geq 3$ and t is odd. □

Theorem 14 could be used in place of Theorem 4, to prove Theorem 2. It would be used in Case 1 of the proof, and it would simplify Case 3 dramatically. However, Theorem 14 has not yet appeared, its proof is very long and complicated, and so its use is avoided in this paper.

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