A Reduction Method for Graphs

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We shall use the notation of Bondy and Murty [2], except that we regard graphs as having no loops. For $k \geq 2$, the 2-regular connected graph of order k is called a <u>k-cycle</u> and is denoted C_k.

For any graph G and any edge $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e and by deleting any resulting loops, which are not allowed. For any connected subgraph H of G, let G/H denote the graph obtained from G by contracting all edges of $E(H)$ and by deleting any resulting loops.

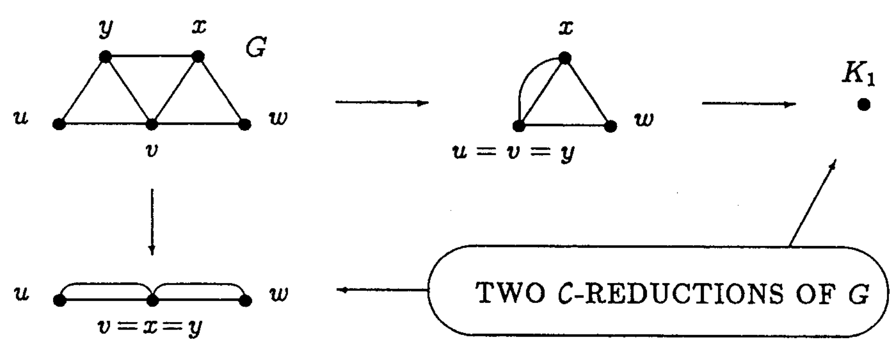
A family of graphs will be called a <u>family</u>. We say that a family $\mathcal{F}$ is <u>closed under contraction</u> if for any $G \in \mathcal{F}$ and any connected subgraph $H \subseteq G$, $G/H \in \mathcal{F}$. The family $\mathcal{F}$ is <u>closed under edge addition</u> if for any graph $G \in \mathcal{F}$ and any distinct vertices $v, w \in V(G)$, the graph $G + vw$, obtained by adding a new edge vw to G is in $\mathcal{F}$. For a graph G and a family $\mathcal{F}$, whenever there is a graph $G' \in \mathcal{F}$ and a set $E' \subseteq E(G')$ such that $G = G' - E'$, we say that G is <u>at most</u> k <u>edges short</u> of being in $\mathcal{F}$.

For any family $\mathcal{C}$ whose members are connected, a $\mathcal{C}$-<u>reduction</u> of G is a graph obtained from G by repeated contractions of subgraphs in $\mathcal{C}$ until the resulting graph has no nontrivial subgraph in $\mathcal{C}$. (The only possible trivial subgraph in $\mathcal{C}$ is K_1). For example, if

$$\mathcal{C} = \{C_3\},$$

then the graph G in Figure 1 has two $\mathcal{C}$-reductions. Since $C_2 \notin \mathcal{C}$, $\mathcal{C}$ is not closed under contraction.

FIGURE 1



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Theorem 1 [11] If a family $\mathcal{C}$ of connected graphs is both closed under contraction and closed under edge addition, then any graph G has a unique $\mathcal{C}$-reduction.

In [11] we give an example to show that it is not sufficient in Theorem 1 merely to assume that $\mathcal{C}$ is closed under contraction.

Let $\mathcal{S}$ and $\mathcal{C}$ be graph families closed under contraction, such that all graphs in $\mathcal{C}$ are connected and $\mathcal{C}$-reductions are unique. In this paper, we shall present interesting examples of such families $\mathcal{S}$ and $\mathcal{C}$ for which the following equivalence holds for every graph G:

$$(1) \qquad G \in \mathcal{S} \iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is in } \mathcal{S}.$$

For most families $\mathcal{S}$, the equivalence (1) holds only if $\mathcal{C} = \{K_1\}$, a trivial case, but we shall present nontrivial instances of (1). Note that if $\mathcal{S} = \mathcal{C}$ in (1), then we have

$$(2) \qquad G \in \mathcal{C} \iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is in } \mathcal{C}$$
$$\iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is } K_1.$$

EXAMPLE 1: AN ILLUSTRATION

Suppose that $\mathcal{C}$ is the family of 2-edge-connected graphs, and let G' denote the $\mathcal{C}$-reduction of G. For any graph G, the set of cut-edges of G is the set of edges of G', and G' is a forest. If $\mathcal{S}$ is the family of graphs with exactly three cut-edges, say, then (1) is easily verified.

EXAMPLE 2: SPANNING CLOSED TRIALS

Define, for any graph H,

$$O(H) = \{v \in V(H) \mid d_H(v) \text{ is odd}\}.$$

A graph H is called eulerian if H is connected and $O(H) = \emptyset$. We call a graph supereulerian if it has a spanning eulerian subgraph. By Euler's Theorem ([2], p. 51) $H \in \mathcal{SL}$ if and only if H has a spanning closed trail. Denote the family of supereulerian graphs by $\mathcal{SL}$. Of course, $K_1 \in \mathcal{SL}$.

A graph G is called collapsible if for any even subset $X \subseteq V(G)$ there is a spanning connected subgraph Γ_X of G such that $O(\Gamma_X) = X$. We denote the family of collapsible graphs by $\mathcal{CL}$. Since X may be the empty set in that definition, we have:

$$\mathcal{CL} \subset \mathcal{SL}.$$

In [3], we proved that any graph G has a unique $\mathcal{CL}$-reduction, and that

(3) $$G \in \mathcal{SL} \iff \text{The } \mathcal{CL}\text{-reduction of } G \text{ is in } \mathcal{SL},$$

and

(4) $$\begin{aligned} G \in \mathcal{CL} &\iff \text{The } \mathcal{CL}\text{-reduction of } G \text{ is in } \mathcal{CL} \\ &\iff \text{The } \mathcal{CL}\text{-reduction of } G \text{ is } K_1. \end{aligned}$$

We also proved

Theorem 2 [3] Let G be a graph. If G is at most one edge short of having two edge-disjoint spanning trees, then $G \in \mathcal{CL}$ or G has a cut-edge.

Corollary 2A (Jaeger [18]) If G has two edge-disjoint spanning trees, then $G \in \mathcal{SL}$.

Corollary 2B The 2-cycle and 3-cycle are collapsible.

The graph $G = K_{2,t}$ $(t \geq 1)$ is two edges short of having two edge-disjoint spanning trees. It can be checked that $K_{2,t} \notin \mathcal{CL}$ by letting X (in the definition of $\mathcal{CL}$) be the two nonadjacent vertices of degree t in $K_{2,t}$. We have conjectured ([3] and [11]) that the only connected graphs not in $\mathcal{CL}$ that are at most two edges short of having two edge disjoint spanning trees are contractible either to $K_{2,t}$ $(t \geq 1)$ or to K_2. Any graph with a cut-edge is not collapsible.

The statement (3) follows from the following:

Theorem 3 [3] If H is a graph in $\mathcal{CL}$, then for any graph G having H as a subgraph,

(5) $$G \in \mathcal{SL} \iff G/H \in \mathcal{SL}.$$

That (3) is a consequence of Theorem 3 follows from the fact [3] that in any graph G there is a unique set of maximal collapsible subgraphs $H_1, H_2, ..., H_c$, and they are pairwise disjoint. (Note that G is the $\mathcal{CL}$-reduction of itself if and only if each maximal collapsible subgraph H_i is just K_1.) Then (3) is obtained by applying Theorem 3 to each H_i $(1 \leq i \leq c)$.

The reductions (3) and (4) have been applied by P. A. Catlin and H.-J. Lai to study $\mathcal{SL}$, $\mathcal{CL}$, and hamiltonian line graphs in a series of other papers ([4], [5], [6], [7], [8], [12], [20], [21], [22], [23]).

EXAMPLE 3: DOUBLE CYCLE COVERS

An instance of the equivalence (1) can be applied to the study of double cycle covers. It is trivial that if G is a planar graph with no cut-edges, then G has a family $\mathcal{F}$ of cycles, such that each edge of G lies in exactly two of the cycles of $\mathcal{F}$: let $\mathcal{F}$ be the family of cycles of G that form the facial boundaries in a planar embedding of G. It has been conjectured (for a survey, see [19]) that the hypothesis of planarity can be omitted:

Double Cycle Cover Conjecture If a graph G has no cut-edge, then G has a family $\mathcal{F}$ of cycles such that each edge of G lies in exactly two of the cycles of $\mathcal{F}$.

Call a graph even if it has no vertex of odd degree. By Euler's Theorem [2], a graph is even if and only if it is an edge-disjoint union of cycles. Thus, we can restate the Double Cycle Cover Conjecture in an equivalent form:

Double Cycle Cover Conjecture, restated: If a graph G has no cut-edge, then G has a family $\mathcal{E}$ of even subgraphs, such that every edge of G lies in exactly two members of $\mathcal{E}$.

For a given graph G having no cut-edge, what is the smallest cardinality of a family $\mathcal{E}$ of even subgraphs of G, such that the restated conjecture holds? Clearly, the smallest such $\mathcal{E}$ has $|\mathcal{E}| = 2$, and this occurs whenever G is an even graph, for $\mathcal{E}$ then consists of two copies of G. It is easy to show that any supereulerian graph has such a family $\mathcal{E}$ with $|\mathcal{E}| \leq 3$. Bermond, Jackson, and Jaeger ([1], p. 302) showed that if a graph G has a family $\mathcal{E}$ with $|\mathcal{E}| = 4$ that satisfies the restated conjecture, then there is also a family $\mathcal{E}'$ of three even subgraphs of G that also satisfies the conjecture. For the Petersen graph, the smallest family $\mathcal{E}$ is a family of five even graphs (all cycles) that together form the desired double cover. Tarsi [30] showed that if a graph G with no cut-edge has a hamiltonian path, then G has a double cover $\mathcal{E}$ with $|\mathcal{E}| \leq 6$ in the restated conjecture.

Define $\mathcal{S}_k$ $(k \geq 2)$ to be the family of graphs G without cut-edges such that G has a double cover $\mathcal{E}$ of even graphs with $|\mathcal{E}| \leq k$. Thus,

$$\{\text{even graphs}\} = \mathcal{S}_2 \subset \mathcal{S}_3 = \mathcal{S}_4 \subset \mathcal{S}_5,$$

and it has been conjectured that $\mathcal{S}_5 = \{\text{graphs with no cut-edges}\}$. Also,

$$\mathcal{SL} \subset \mathcal{S}_3 \cap \{\text{connected graphs}\}.$$

It is easy to check that if G is 3-regular, then

$$G \in \mathcal{S}_3 \iff \chi'(G) = 3.$$

The following conjecture was made by Tutte [34] for 3-regular graphs, and by Matthews [25] in its present form:

Conjecture (Tutte-Matthews) If G is a graph, then at least one of the following holds:

(a) $G \in \mathcal{S}_3$;

(b) G has a cut-edge;

(c) G has a subgraph contractible to the Petersen graph.

It is easy to check that (a) and (b) are mutually exclusive. We proved a related result:

Theorem 4 [10] If a graph G is at most 5 edges short of being 4-edge-connected, then exactly one of the following holds:

(a) $G \in \mathcal{S}_3$;

(b) G has a cut-edge;

(c) G is contractible to the Petersen graph.

A significant portion of the proof of Theorem 4 was a demonstration of the following equivalence of the form of (1):

(6) $$G \in \mathcal{S}_3 \iff \text{The } (\mathcal{CL} \cup \{C_4\})\text{-reduction of } G \text{ is in } \mathcal{S}_3.$$

Since $\mathcal{CL} \cup \{C_4\}$ contains all cycles of length at most 4, it follows from the definition of reduction that the $(\mathcal{CL} \cup \{C_4\})$-reduction of G has girth at least 5.

EXAMPLE 4: PACKING SPANNING TREES

Define the invariant

(7) $$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{\omega(G-E)-1},$$

where E is chosen so that $\omega(G-E)$, the number of components of $G-E$, is greater than 1. Cunningham [13] proved

Theorem 5 Let G be a graph, and let $s, t \in \mathbf{N}$. These are equivalent:
(a) $\eta(G) \geq s/t$;
(b) G has a family $\mathcal{F}$ of s spanning trees such that each edge of G lies in at most t trees of $\mathcal{F}$.

The case $t = 1$ of Theorem 5 is due to Tutte [33] and Nash-Williams [26]. It asserts that $\lfloor \eta(G) \rfloor$ is the maximum number of edge-disjoint spanning trees in G.

In [9], it was noted essentially that if

(8) $$\mathcal{C} = \{G \mid \eta(G) \geq r\} \cup \{K_1\},$$

where $r \geq 1$ is real, then

(9) $$\begin{aligned} G \in \mathcal{C} &\iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is in } \mathcal{C} \\ &\iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is } K_1. \end{aligned}$$

EXAMPLE 5: EDGE-CONNECTIVITY

Let $\kappa'(G)$ be the edge-connectivity of G. For $k \in \mathbf{N}$, define

$$\mathcal{C} = \{G \mid \kappa'(G) \geq k\} \cup \{K_1\}.$$

Then

$$G \in \mathcal{C} \iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is in } \mathcal{C}$$
$$\iff \text{The } \mathcal{C}\text{-reduction of } G \text{ is } K_1.$$

See [9] for more details. Mader [23] gave a different and powerful reduction method for edge-connectivity, and it has also been applied to problems involving Example 6 (see e.g., [27]) and to the proof of Theorem 4 [10]. We [9] obtained the following relation between η and κ' that improves upon the widely known result (a corollary of the Tutte [33] and Nash-Williams [26] Theorem) that a $2k$-edge-connected graph has k edge-disjoint spanning trees. S.-M. Zhan [35] had proved the "$\Longrightarrow$" part of the case $k = 2$ of this result:

Theorem 6 [9] Let $k \in \mathbf{N}$, let G be a graph, and let $\mathcal{E}_k$ be the collection of all k-element subsets of $E(G)$. Then

$$\kappa'(G) \geq 2k \iff \text{For any } E \in \mathcal{E}_k,\ \eta(G - E) \geq k;$$

and

$$\kappa'(G) \geq 2k + 1 \iff \text{For any } E \in \mathcal{E}_k,\ \eta(G - E) > k.$$

EXAMPLE 6: EDGE-DISJOINT PATHS

Let $k \in \mathbf{N}$. Let $\mathcal{C}_k$ be the family of graphs G satisfying the following condition:

For any $2k$ vertices $s_1, t_1, s_2, t_2, ..., s_k, t_k \in V(G)$ (not necessarily distinct) there are pairwise disjoint (s_i, t_i)-paths P_i $(1 \leq i \leq k)$.

Trivially, $K_1 \in \mathcal{C}_k$ for any $k \in \mathbf{N}$. Note, for example, that the 4-cycle is not in $\mathcal{C}_2$, for its distinct vertices may be labelled s_1, s_2, t_1, t_2, consecutively. In the literature, graphs in $\mathcal{C}_k$ are called weakly k-linked. Seymour [28] and Thomassen [31] have characterized $\mathcal{C}_2$ by characterizing an infinite family $\mathcal{F}_2$ (say) of graphs (including the 4-cycle) such that any graph not in $\mathcal{C}_2$ is contractible to a member of $\mathcal{F}_2$. Frank [15] studied $\mathcal{C}_k$ for planar graphs.

Conjecture (Thomassen [32]) Let $k \in \mathbf{N}$ and suppose that G is a k-edge-connected graph. Then $G \in \mathcal{C}_k$ if k is odd, and $G \in \mathcal{C}_{k-1}$ if k is even.

Okamura [27], Cypher [14], Enomoto and Saito [15], and Hirata, Kubota and Saito [17] proved this conjecture for small values of k.

Theorem 7 Let $k \in \mathbf{N}$. Let G be a graph and let H be a subgraph of G. If $H \in \mathcal{C}_k$, then

$$G \in \mathcal{C}_k \iff G/H \in \mathcal{C}_k. \tag{10}$$

Proof: Suppose that k, G, and H satisfy the hypothesis of Theorem 7.

Proof of "$\Longrightarrow$": Suppose $G \in \mathcal{C}_k$, and let $u_1, v_1, u_2, v_2, ..., u_k, v_k$ be $2k$ vertices of G/H, not necessarily distinct. Let v_H denote the vertex of G/H corresponding to H, i.e., the vertex onto which H gets contracted. Let the $2k$ vertices $s_1, t_1, s_2, t_2, ..., s_k, t_k \in V(G)$ satisfy

$$\begin{aligned} s_i &= u_i && \text{if } u_i \neq v_H; \\ s_i &\in V(H) && \text{if } u_i = v_H; \\ t_i &= v_i && \text{if } v_i \neq v_H; \\ t_i &\in V(H) && \text{if } v_i = v_H. \end{aligned}$$

Since $G \in \mathcal{C}_k$, there are pairwise edge-disjoint (s_i, t_i)-paths P_i $(1 \leq i \leq k)$. For each i, let $R_i = P_i$ if $V(H) \cap V(P_i) = \emptyset$; but if $V(H) \cap V(P_i) \neq \emptyset$, then let R_i be the subgraph of P_i consisting of the union of the subpath of P_i from s_i to the first vertex of P_i in $V(H)$ and the subpath of P_i from the last vertex of P_i in $V(H)$ to t_i. Then in G/H, each R_i induces a (u_i, v_i)-path. Also, since the P_i's are edge-disjoint, so are the paths in G/H induced by the R_i's. Hence, $G/H \in \mathcal{C}_k$.

Proof of "$\Longleftarrow$": Suppose that $G/H \in \mathcal{C}_k$ and let $s_1, t_1, s_2, t_2, ..., s_k, t_k$ be $2k$ vertices of G. As before, let v_H be the vertex of G/H corresponding to H. In G/H, define the vertices

$$s_i' = \begin{cases} s_i & \text{if } \; s_i \notin V(H), \\ v_H & \text{if } \; s_i \in V(H); \end{cases}$$

$$t_i' = \begin{cases} t_i & \text{if } \; t_i \notin V(H) \\ v_H & \text{if } \; t_i \in V(H). \end{cases}$$

Since $G/H \in \mathcal{C}_k$, there are k edge-disjoint paths $Q_1', Q_2', ..., Q_k'$ in G/H, such that Q_i' has ends s_i' and t_i' $(1 \leq i \leq k)$. Let $I \subseteq \{1, 2, ..., k\}$ be the indices such that

$v_H \in V(Q_i')$. If $i \notin I$, then Q_i' induces an (s_i, t_i)-path P_i (say) in G. But if $i \in I$ then let Q_i denote the union of these two paths: the path in G from s_i to some $u_i \in V(H)$ induced by the (s_i', v_H)-segment of Q_i'; and the path in G from some $v_i \in V(H)$ to t_i induced by the (v_H, t_i')-segment of Q_i'.

Since $H \in \mathcal{C}_k$, there are $|I| \leq k$ pairwise edge-disjoint (u_i, v_i)-paths R_i (say) in H, where $i \in I$. If $i \in I$, then define $P_i = Q_i \cup R_i$; recall that P_i was already defined when $i \notin I$. Then P_i is an (s_i, t_i)-path in G $(1 \leq i \leq k)$, and the P_i's are pairwise edge-disjoint. Hence, $G \in \mathcal{C}_k$. □

Corollary For any $k \in \mathbb{N}$, if G is a graph, then

$$(11) \qquad G \in \mathcal{C}_k \iff \text{The } \mathcal{C}_k\text{-reduction of } G \text{ is in } \mathcal{C}_k$$
$$\iff \text{The } \mathcal{C}_k\text{-reduction of } G \text{ is } K_1.$$

Proof: As with (2), the first equivalence of (11) implies the second. Let G' be the $\mathcal{C}_k$-reduction of G. By definition, G' is obtained from G by a sequence of contractions of subgraphs $H' \in \mathcal{C}_k$. Apply (10) at each step in this sequence to show that the intermediate contractions of G are in $\mathcal{C}_k$. Hence, (10) implies the first part of (11). □

GENERAL REMARKS

The equivalence (1) becomes more powerful if $\mathcal{C}$ can be shown to be a large family, because a larger family $\mathcal{C}$ yields a smaller family of graphs that are $\mathcal{C}$-reductions. We conjectured ([9] and [10]) that when $\mathcal{S} = \mathcal{SL}$ in (1) (Example 2), $\mathcal{C} = \mathcal{CL}$ is the largest possible value of $\mathcal{C}$. In other words, we conjectured that (3) is best-possible.

Let $\mathcal{C}$ be a family of connected graphs satisfying (2). Since the $\mathcal{C}$-reduction G' of any graph G has no subgraph in $\mathcal{C}$, we can ask the extremal question: given a graph G of order n that has no nontrivial subgraph in $\mathcal{C}$ (i.e., that is the $\mathcal{C}$-reduction of itself), what is the maximum possible number of edges of G? By Example 1, if $\mathcal{C} = \{$2-edge-connected graphs$\}$, then a maximal such graph is a tree, and hence has $n-1$ edges. Let $r > 1$. If $\mathcal{C}$ satisfies (8) of Example 4, then any $\mathcal{C}$-reduction of order n has arboricity less than r, and so a maximal such $\mathcal{C}$-reduction of order n has fewer than $r(n-1)$ edges. Since the family $\mathcal{C}$ contains no tree or forest (because $r > 1$), $\mathcal{C}$ is a counterexample to Theorem 6.7 (page 181) of [29]. (Theorem 6.7 of [29] is valid if $|\mathcal{L}|$ is finite, but not if $\mathcal{L} = \mathcal{C}$ of (8)).

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