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Graph Homomorphisms into the Five-Cycle

PAUL A. CATLIN

Department of Mathematics, Wayne State University, Detroit, Michigan 48202

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We consider those edge-minimal graphs having no homomorphism into the five-cycle. We characterize constructively such graphs having the additional property that they contain no topological K_4 as a subgraph. © 1988 Academic Press, Inc.

1. INTRODUCTION

For simple graphs G and H, we consider the graph homomorphism

$$\theta: G \to H, \tag{1}$$

where θ maps $V(G)$ into $V(H)$ and where $xy \in E(G)$ implies $\theta(x)\,\theta(y) \in E(H)$. When H is a complete graph, the homomorphism θ is the usual coloring, and the chromatic number and achromatic number are special cases. (These numbers and homomorphisms are related by the Homomorphism Interpolation Theorem [7]. For a bound, see [3].)

When the homomorphism (1) exists, we shall call θ an *H-coloring* of G. If G has an H-coloring, then we call G *H-colorable*. If G has no H-coloring, but for all $e \in E(G)$, $G-e$ has an H-coloring, we say that G is *H-critical*. For example, a graph is K_{n-1}-critical in this sense if and only if it is chromatically n-critical in the usual sense (of [2], for example).

A graph F is *uniquely H-colorable* if for any H-colorings θ_1 and θ_2 of F there is an automorphism $\varnothing$ of H such that $\varnothing\theta_1 = \theta_2$.

PROPOSITION 1. *If G is H-critical, then G cannot be separated by a uniquely H-colorable subgraph F.*

The proof is an imitation of the proof for the case $H = K_n$, i.e., for chromatically critical graphs. We omit the details.

An (x, y)-*arc* $A(x, y)$ of G is a maximal path in G whose ends are x, $y \in V(G)$ and whose interval vertices are divalent in G. Either x and y

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are not divalent, or $x=y$ and the component of G containing x is a cycle. An (x, y)-arc $A(x, y)$ having n edges will be denoted $A_n(x, y)$. If $A_n(x, y)$ is an arc of G, then $G-A_n(x, y)$ will denote the subgraph of G obtained by removing all edges and *internal* vertices of $A_n(x, y)$.

PROPOSITION 2. *If G is C_{2k+1}-critical, then no arc of G has more than $2k-1$ edges.*

Since the proof is routine, we omit it. We shall refer to both propositions in the next section.

We define an *odd-TK_4* to be a TK_4 which, when embedded in the plane, has all four faces of odd girth. An *odd-K_3^2* is defined to be any graph consisting of three edge-disjoint odd cycles C, C', C'', and three arcs

$$A(u, u') \quad (u \in V(C), \quad u' \in V(C')),$$
$$A(v', v'') \quad (v' \in V(C'), \quad v'' \in V(C'')),$$
$$A(w'', w) \quad (w'' \in V(C''), \quad w \in V(C)),$$

whose internal vertices have degree 2. (The graph R of Fig. 1 is an example of an odd-K_3^2 in which all three arcs have length 0.)

Dirac [5] proved that if a graph has no C_3-coloring, then it has a TK_4. We [4] showed that the TK_4 in the conclusion of Dirac's theorem could be chosen to be an odd-TK_4. Gerards [6], in strengthening a result of [1], proved the following result:

THEOREM 1. *Let G be a graph with odd girth $2k+1$. Either G has a C_{2k+1}-coloring, or G contains an odd-TK_4 or an odd-K_3^2.*

In this paper, we shall characterize constructively the graphs with no C_5-coloring and no TK_4 subgraph.

2. THE MAIN RESULTS

The *branch graph* $B(G)$ of a graph G (G not a cycle) is the multigraph obtained from G by replacing every arc by an edge joining its ends. A graph is *nodally 3-connected* if its branch graph is 3-connected (this is equivalent to Tutte's definition [8]). For an induced subgraph H of G, the *vertices of attachment* of H in G are those vertices of H incident with at least one edge of $E(G)-E(H)$.

We use $d(u, v)$ to denote the distance in C_5 between $u, v \in V(C_5)$.

For $x, y \in V(H)$, define

$$D(x, y, H)=\{d(\theta(x), \theta(y)) \mid \theta \text{ is a } C_5\text{-coloring of } H\}.$$

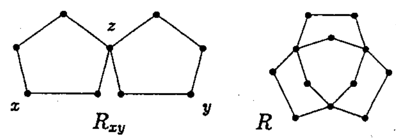

FIGURE 1

Of course, θ runs over all C_5-colorings of H. Thus,

$$D(x, y, H) \subseteq \{0, 1, 2\}.$$

Given two copies C, C' of C_5, with distinguished vertices x, $z \in V(C)$ at distance 2 in C, and with distinguished vertices y, $z' \in V(C')$ at distance 2 in C', we denote by R_{xy} the nine-vertex graph obtained from $C \cup C'$ by identifying $z = z'$. See Fig. 1.

We shall denote by $H + A_n(x, y)$ the graph obtained by adding to H an (x, y)-arc $A_n(x, y)$ having n edges, where x, $y \in V(H)$. Denote (see Figs. 1 and 2)

$$R'(x, y) = R_{xy} + A_2(x, y),$$
$$R''(x, y) = R_{xy} + A_3(x, y),$$
$$R = R'(x, y) + A_3(x, y),$$

and

$$R_0(x, v) = R_{xy} + A_5(y, y), \qquad v \in V(A_5(y, y)), \qquad d(v, y) = 2.$$

Thus, $R_0(x, v)$ consists of three blocks, each a 5-cycle, and x, y, v are distinguished vertices, with y as a cutvertex.

An *incremental subgraph* H of a graph G is an induced subgraph H either isomorphic to $R'(x, y)$ or $R''(x, y)$ and with vertices of attachment $\{x, y\}$ in G, or isomorphic to $R_0(x, v)$, with vertices of attachment $\{x, v\}$ in G, where $v \in V(A_5(y, y)) \subseteq V(R_0(x, v))$ is at distance 2 from y.

THEOREM 2. *If G is a C_5-critical graph with no TK_4 subgraph, and if G is neither K_3 nor R, then G contains two edge-disjoint incremental subgraphs.*

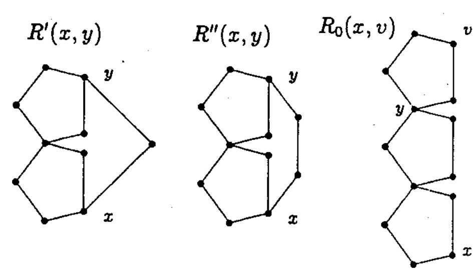

FIG. 2. The incremental subgraphs.

Proof. Throughout this proof, G will denote a C_5-critical graph, neither K_3 nor R, and without a TK_4.

Suppose that G is nodally 3-connected. Then the underlying branch graph $B(G)$ is 3-connected. Therefore, there are vertices x, y of degree at least 3 in G, and there are three internally disjoint (x, y)-paths P_1, P_2, P_3 in G, by Menger's theorem. Also, $B(G)-\{x, y\}$ is connected, and hence some path P_4 in G joins internal vertices of two of P_1, P_2, P_3. Then $P_1 \cup P_2 \cup P_3 \cup P_4$ is a TK_4 subgraph of G.

Hence, by Proposition 1, we can assume that G has connectivity and nodal connectivity 2. We shall also suppose inductively, for the remainder of the proof, that for any C_5-critical graph $G' \notin \{K_3, R\}$, with $|V(G')| < |V(G)|$, where G' has no TK_4, there are two edge-disjoint incremental subgraphs in G'. As a basis for induction, note that if $|V(G)| \leqslant 5$, then the induction hypothesis holds vacuously. ∎

We shall prove some lemmas next. In these lemmas, unions and intersections are defined as in [2].

LEMMA 1. *If G_{xy} is a 2-connected subgraph of G with vertices of attachment $\{x, y\}$ in G, where $G_{xy} \neq K_2$, then G_{xy} can be decomposed into connected subgraphs H, H' such that*

$$H \cup H' = G_{xy}, \qquad H \cap H' = \{x, y\}.$$

Proof. Since G_{xy} is 2-connected with vertices of attachment $\{x, y\}$ in G, there are internally disjoint (x, y)-paths P, P' in G_{xy}. Since G is 2-connected, $G - E(G_{xy})$ has an (x, y)-path P_0. If a path P'' in G_{xy}-$\{x, y\}$ joins an internal vertex of P to an internal vertex of P', then $P_0 \cup P \cup P' \cup P''$ is a TK_4 in G, contrary to the hypothesis of the theorem. Hence, no such path P'' exists, and so $\{x, y\}$ separates G_{xy}, and subgraphs H and H' exist as described, where $P \subseteq H$, $P' \subseteq H'$. ∎

LEMMA 2. *An acyclic subgraph of G with only two vertices of attachment $\{u, v\}$ in G is a (u, v)-path.*

Proof. An acyclic subgraph H of G is a tree. Since G has no cutvertex (by Proposition 1), each vertex of degree 1 in H is a vertex of attachment in G. Since H has only two vertices of attachment (u and v) in G, H must be a (u, v)-path. ∎

LEMMA 3. *There exist $x, y \in V(G)$ and connected subgraphs H_1 and H_2 of G, such that*

$$G = H_1 \cup H_2, \qquad \{x, y\} = H_1 \cap H_2, \tag{2}$$

and such that

$$H_1 \text{ and } H_2 \text{ each contain at least one cycle.} \tag{3}$$

Proof. Since the nodal connectivity of G is 2, and since G is not a cycle, the underlying branch graph $B(G)$ has a separating set $\{x, y\}$. Therefore, connected subgraphs H_1 and H_2, satisfying (2), exist, where H_1 and H_2 both have vertices of degree at least 3 and different from x and y. If H_i has no cycle, then by Lemma 2, H_i is an (x, y)-path, a contradiction. Therefore, H_1 and H_2 each contain a cycle. ▮

Since G is C_5-critical, some graph H_i ($i \in \{1, 2\}$) of Lemma 3 satisfies $|D(x, y, H_i)| = 1$, and so we lose no generality in assuming that

$$|D(x, y, H_1)| = 1 \tag{4}$$

and

$$H_1 \text{ is maximal with respect to (2), (3), and (4).} \tag{5}$$

Any ordered pair (H_1, H_2) of induced subgraphs of G satisfying (5) (and hence (2), (3), and (4)) for some separating set $\{x, y\}$ will be called a *proper pair* of subgraphs of G.

Clearly, for $i \in \{1, 2\}$, since G is 2-connected,

$$\text{All cutvertices of } H_i \text{ lie on a single } (x, y)\text{-path.} \tag{6}$$

Let H_0 be a 2-connected induced subgraph of G with vertices of attachment $\{u, v\}$ in G. If

$$D(u, v, H_0) = \{0\},$$

then H_0 is called a *zero*-block.

LEMMA 4. *If (H_1, H_2) is a proper pair, then H_2 has no zero-block.*

Proof. Suppose that H_{uv} is a zero-block of H_2. By the definition of a zero-block,

$$D(u, v, H_{uv}) = \{0\}. \tag{7}$$

By Lemma 1, H_{uv} has subgraphs H, H' such that

$$H_{uv} = H \cup H', \qquad \{u, v\} = H \cap H'.$$

Since G is C_5-critical, (7) implies

$$D(u, v, G - (H_{uv} - \{u, v\})) \subseteq \{1, 2\},$$

and the values of $D(u, v, G-(H-\{u, v\}))$ and $D(u, v, G-(H'-\{u, v\}))$ are $\{1\}$ and $\{2\}$ in some order. Hence, H or H' could have been chosen in place of H_2 in (2) and (3), unless both H and H' are acyclic. This contradicts the maximality of H_1 in (4) and (5), except when both H and H' are acyclic. In the latter case, by Lemma 2, they are (u, v)-paths, and thus H_{uv} is a cycle. But then (7) is false, a contradiction. ∎

LEMMA 5. *If (H_1, H_2) is a proper pair, then H_2 has a cutvertex.*

Proof. Suppose, by way of contradiction, that H_2 is 2-connected. By (3), $H_2 \neq K_2$. By Lemma 1, there are subgraphs H, H' of H_2, such that

$$H_2 = H \cup H', \qquad \{x, y\} = H \cap H'.$$

Since G is C_5-critical and $D(x, y, H_1)$ is a singleton (by (4)), we have identical singletons

$$D(x, y, H_1 \cup H) = D(x, y, H_1 \cup H').$$

But this implies that G has a C_5-coloring, a contradiction. Therefore, H_2 has at least one cutvertex. ∎

LEMMA 6. *If (H_1, H_2) is a proper pair, then $H_2 = R_{xy}$ and $D(x, y, H_1) = \{2\}$.*

Proof. Let H_x (resp., H_y) be the block of H_2 containing x (resp., y). By Lemma 5, $H_x \neq H_y$. Let $\{x, x'\}$ (resp., $\{y, y'\}$) be the vertices of attachment of H_x (resp., of H_y) in G.

If

$$|D(x, x', H_x)| = |D(y, y', H_y)| = 2,$$

then G has a C_5-coloring, a contradiction. Hence, there is no loss of generality in our supposing that

$$D(x, x', H_x) = \{t\}, \tag{8}$$

and Lemma 4 implies $t \in \{1, 2\}$. By (4), $|D(x, y, H_1)| = 1$.

Case 1. Suppose $D(x, y, H_1) = \{0\}$. Define $H_1' = H_1 \cup H_x$, and note that (8) implies

$$D(x, y, H_1 \cup H_x) = \{t\}.$$

By the maximality of H_1 in (5), the induced subgraph $H_2' = H_2 - (H_x - x')$ is acyclic with vertices of attachment $\{x', y\}$, and so by Lemma 2, H_2' is an (x', y)-path P. Let $yz \in E(P)$ be the edge incident with y. Then

$D(x, z, H_1 \cup yz) = \{1\}$, and the graph induced by $H_1 \cup yz$ contradicts the maximality of H_1 in (5).

Case 2. Suppose $D(x, y, H_1)$ is $\{1\}$ or $\{2\}$. By (6), the blocks of H_1 and the blocks of H_2 are arranged in cyclic order in G. By Lemma 4 and the condition of Case 2, if H_2 has two or more cutvertices (and hence at least three blocks), then G has a C_5-coloring. Hence, H_2 has a unique cutvertex $z = x' = y'$.

Since G has no C_5-coloring, there is no triple $a_1, a_x, a_y \in \{1, 2\}$ with

$$a_1 \in D(x, y, H_1), \qquad a_x \in D(x, z, H_x), \qquad a_y \in D(y, z, H_y)$$

such that for some choice of plus and minus signs, chosen independently,

$$a_1 \pm a_x \pm a_y \equiv 0 \pmod{5}. \tag{9}$$

The absence of zero-blocks implies $0 \notin \{a_1, a_x, a_y\}$. If any of $D(x, y, H_1)$, $D(x, z, H_x)$, $D(y, z, H_y)$ has more than one member, then (9) has a solution, a contradiction. If all three sets have exactly one member, then since (9) has no solution, all are $\{1\}$ or all are $\{2\}$.

Suppose that for some $k \in \{1, 2\}$, we have $k = a_1$, $k = a_x$, $k = a_y$. Then

$$D(x, z, H_1 \cup H_y) = \{0, 3 - k\}. \tag{10}$$

By Lemma 1, the block H_x can be decomposed into connected subgraphs H, H', where

$$H_x = H \cup H', \qquad \{x, z\} = H \cap H'.$$

Since G is C_5-critical, it follows from (10) that $D(x, z, H_1 \cup H_y \cup H)$ and $D(x, z, H_1 \cup H_y \cup H')$ are $\{0\}$ and $\{3 - k\}$ in some order.

If H' contains a cycle, then $H_1 \cup H_y \cup H$ would violate the maximality of H_1 in (5), since H' could replace H_2 in (3). Therefore, H' is acyclic. Likewise, H is acyclic. By Lemma 2, both H and H' are (x, z)-paths. Since G is C_5-critical, the lengths of H and H' are less than four, by Proposition 2, and they are unequal. By Proposition 1, x and z are not adjacent. Hence, one of H, H' has length 2 and the other has length 3, and so H_x is a 5-cycle, and $k = 2$.

A similar argument shows that H_y is a 5-cycle, with (y, z)-arcs of lengths 2 and 3. Therefore, $H_2 = H_x \cup H_y = R_{xy}$, and $D(x, y, H_1)$ must be $\{2\}$. Lemma 6 is proved. ▌

Lemma 7. *If (H_1, H_2) is a proper pair of subgraphs of G, and if H_1 is 2-connected, then either Theorem 2 holds for G or there are subgraphs H, H' of H_1 such that*

$$H_1 = H \cup H', \qquad \{x, y\} = H \cap H', \qquad H = A_2(x, y),$$

where $\{x, y\}$ is the set of vertices of attachment of H, of H', and H_2 in G.

Proof. By Lemma 1, since H_1 is 2-connected, there are subgraphs H, H' such that

$$H_1 = H \cup H', \qquad \{x, y\} = H \cap H'.$$

Since G is C_5-critical, the three sets $D(x, y, H_2)$, $D(x, y, H)$, and $D(x, y, H')$ are distinct subsets of $\{0, 1, 2\}$ such that none of the three sets contains another one of the three sets. By Lemma 6,

$$D(x, y, H_2) = \{0, 1\},$$

and so we lose no generality in supposing

$$D(x, y, H) = \{0, 2\}, \qquad D(x, y, H') = \{1, 2\}.$$

Therefore, $H + A_1(x, y)$ is C_5-critical and has no TK_4, and by the induction hypothesis, either $H + A_1(x, y) = K_3$, whence $H = A_2(x, y)$, as required by Lemma 7, or H contains an incremental subgraph F of G. It remains to exclude the latter case.

Suppose that H has an incremental subgraph F. Let G' denote the graph obtained from G upon the replacement of H by $A_2(x, y)$. Since G is C_5-critical and $D(x, y, H) = D(x, y, A_2(x, y))$, the smaller graph G' is C_5-critical. By the induction hypothesis, G' has two edge-disjoint incremental subgraphs, or $G' = R$. In the former case, $G' - (A_2(x, y) - x - y)$ has an incremental subgraph F', and so F and F' are two edge-disjoint incremental subgraphs of G. In the latter case, since G' includes both $H_2 = R_{xy}$ (by Lemma 6) and $A_2(x, y)$, we must have $H' = A_3(x, y)$. Hence, $H' \cup H_2 = R''(x, y)$ is an incremental subraph F' of G that is edge-disjoint from F. Thus, if H has an incremental subgraph F, then the theorem holds. ∎

LEMMA 8. *If H_0 is a zero-block of G, with*

$$|V(H_0)| \leqslant |V(G)| - 3,$$

then H_0 contains an incremental subgraph.

Proof. Let $u, v \in V(H_0)$ be the vertices of attachment of H_0. Since H_0 is a zero-block and G is C_5-critical with no TK_4, $H_0 + A_3(u, v)$ is also C_5-critical with no TK_4. Since $|V(H_0)| \leqslant |V(G)| - 3$, the induction hypothesis applies to $H_0 + A_3(u, v)$, and so H_0 contains an incremental subgraph. ∎

Proof of Theorem 2 (*continued*). By (5), there is a proper pair (H_1, H_2) of incremental subgraphs of G, and by Lemma 6,

$$G = H_1 \cup H_2, \qquad \{x, y\} = H_1 \cap H_2,$$

and $H_2 = R_{xy}$ is a pair of 5-cycles with exactly one vertex z in common, where $xz,\ yz \notin E(G)$.

Case 1. Suppose that H_1 is 2-connected. By Lemma 7, there are subgraphs H, H' of H_1 such that

$$H_1 = H \cup H', \qquad \{x, y\} = H \cap H', \qquad H = A_2(x, y).$$

Let t be the number of cutvertices of H', and denote $x = z_0$, $y = z_{t+1}$. By (6), we can let $z_1, z_2, ..., z_t$ denote the t cutvertices of H' as they occur along an (x, y)-path in H'.

Since $H = A_2(x, y)$ and since $H_2 = R_{xy}$, the subgraph $F = H \cup H_2$ is an incremental subgraph $R'(x, y)$ in G.

We denote by $B_0, B_1, ..., B_t$ the $t+1$ blocks of H', where

$$z_i, z_{i+1} \in V(B_i) \qquad (0 \leqslant i \leqslant t).$$

If H' is acyclic, then by Lemma 2, H' is an (x, y)-path and since G is C_5-critical and $D(x, y, H \cup H_2) = \{0\}$, we must have $H' = A_3(x, y)$ and hence $G = R$, contrary to our assumption. Therefore, H' contains a cycle, and since $D(x, y, H \cup H_2) = \{0\}$, we have a proper pair (H_3, H_4) satisfying

$$H \cup H_2 \subseteq H_3, \qquad H_4 \subseteq H', \qquad H_3 \cup H_4 = G, \qquad H_3 \cap H_4 = \{z_j, z_k\},$$

for some j and k with $j < k$. By Lemma 6,

$$H_4 = B_j \cup B_{j+1} = R_{uv}, \qquad \text{for} \quad u = z_j,\ v = z_{j+2} = z_k,$$

and $D(u, v, H_3) = \{2\}$. Therefore, $t \geqslant 2$, and B_i is a 5-cycle for some i such that $1 \leqslant i \leqslant t-1$, and so the proper pair (H_3, H_4) may be chosen so that either

$$H \cup H_2 \cup B_0 \subseteq H_3 \qquad \text{or} \qquad H \cup H_2 \cup B_t \subseteq H_3,$$

without violating the requirement (3) that H_4 contain a cycle. If for some h $(0 \leqslant h \leqslant t)$, B_h is a zero-block, then by Lemma 8 and the existence of F, G has two edge-disjoint incremental subgraphs. Hence, we may assume that no B_h is a zero-block. Consequently, $H_3 = H \cup H_2 \cup B_0$ and $H_3 = H \cup H_2 \cup B_t$ are two possible values of H_3 satisfying (5). By Lemma 6, $H_4 = R_{uv}$, where $\{u, v\}$ is $\{z_0, z_2\}$ or $\{z_{t-1}, z_{t+1}\}$, and $t = 2$, since G is C_5-critical. Hence, $H' = R_0(x, y)$ and F are two incremental subgraphs of G.

Case 2. Suppose that H_1 is not 2-connected. Thus, H_1 has at least one cutvertex $v \notin \{x, y\}$. By (6), all cutvertices of H_1 must lie on a single (x, y)-path in H_1.

If H_1 has at least two zero-blocks, then by Lemma 8, G has two incremental subgraphs. Hence, we can assume that H_1 has at most one zero-block. By (4), at most one block of H_1 is not a zero-block. It follows that H_1 has just a single cutvertex v, and so we shall denote by H_{vx} and H_{vy} the two blocks of H_1, where v and x are the two vertices of attachment of H_{vx} in G, and v and y are the two vertices of attachment of H_{vy} in G. Without loss of generality,

$$D(v, y, H_{vy}) = \{0\}, \tag{11}$$

and so by Lemma 6 and (11),

$$D(v, x, H_{vx}) = D(x, y, H_{vx} \cup H_{vy}) = D(x, y, H_1) = \{2\}.$$

By Lemma 8 and (11), H_{vy} has an incremental subgraph F_1.

Denote by H_5 the graph obtained from H_{vx} by adding R_{vx} and identifying both vertices named v and identifying both named x. Note that H_5 is C_5-critical and H_5 has no TK_4 subgraph. Also, $H_5 \neq K_3$.

If $H_5 = R$, then $H_{vx} = C_5$, and so $F_2 = H_{vx} \cup H_2 = R_0(v, y)$ is an incremental subgraph of G. Then F_1 and F_2 are incremental subgraphs.

Suppose, instead, that $H_5 \neq R$. By the induction hypothesis, H_5 has two edge-disjoint incremental subgraphs, say F_3 and F_4. If either one, say F_3, is contained in H_{vx}, then F_1 and F_3 are two edge-disjoint incremental subgraphs of G. If neither F_3 nor F_4 is contained in H_{vx}, then F_3 and F_4 are R_0-type incremental subgraphs of H_5, but since H_{xv} is one block, this is a contradiction.

Therefore, G has two incremental subgraphs, and the induction is complete. Theorem 2 is proved. ∎

THEOREM 3. *The graph G is C_5-critical and has no TK_4 if and only if G is obtained from K_3 by repeated applications of the following three operations:*

1. *The replacement of an arc $A_3(x, v)$ by $R_0(x, v)$ (where x, v of the graph are identified with the corresponding distinguished vertices x, v of $R_0(x, v)$).*
2. *The replacement of an edge xy by $R''(x, y)$.*
3. *The replacement of vertex u by nonadjacent vertices x, y, the joining of every neighbor of u to exactly one of x, y, and the addition of $R'(x, y)$ such that no TK_4 subgraph is created.*

In operations 2 *and* 3, *the distinguished vertices x, y of $R''(x, y)$ or $R'(x, y)$ are identified with the corresponding vertices with the same label in the graph.*

EXAMPLE. The graph R can be obtained from K_3 by a single application of any one of these three operations. See Fig. 1 and 2.

Proof of Theorem 3. By Theorem 2, if G is C_5-critical and has no TK_4 subgraph, then G has an incremental subgraph $R_0(x, v)$, $R'(x, y)$, or $R''(x, y)$. By reversing one of the operations of Theorem 3 on this incremental subgraph, we obtain another TK_4-free C_5-critical graph G' with 9 fewer vertices and 12 fewer edges. By Theorem 2, G can be thus reduced to R and K_3, and so inductively we have

$$|V(G)| \equiv 3 \pmod 9 \tag{12}$$

and

$$3\,|E(G)| + 3 = 4\,|V(G)|. \tag{13}$$

Conversely, let G' be a C_5-critical graph with no TK_4 subgraph. Then G' satisfies (12) and (13). Moreover, one can prove inductively that $D(x, y, G' - xy) = \{0, 2\}$ for all edges xy of G'. Let G be a graph obtained from G' by one of the three operations of the theorem. Clearly, G has no TK_4, and so it remains to show that G is C_5-critical.

Operation 1 replaces a subgraph $A_3(x, v)$ satisfying

$$D(x, v, A_3(x, v)) = \{1, 2\}$$

with the subgraph $R_0(x, v)$ having the property

$$D(x, v, R_0(x, v)) = \{1, 2\},$$

and since G' is C_5-critical, so is G. Operation 2 replaces the edge-subgraph xy, satisfying

$$D(x, y, xy) = \{1\},$$

with the larger subgraph $R''(x, y)$, such that

$$D(x, y, R''(x, y)) = \{1\}.$$

We claim that the graph G resulting from operation 2 is also C_5-critical. By the above remark,

$$D(x, y, G' - xy) = \{0, 2\},$$

and for any proper spanning subgraph H of $G' - xy$, $1 \in D(x, y, H)$, and so $G[E(H) \cup E(R''(x, y))]$ has a C_5-coloring. Also, if $e \in E(R''(x, y))$, then $G - e$ has a C_5-coloring. Thus, G is C_5-critical, as claimed.

Let G be obtained from a C_5-critical graph G' by operation 3. Let

$$G_{xy} = G - (R'(x, y) - \{x, y\}).$$

In Operation 3 we replace a vertex $u \in V(G')$, where $D(u, u, u) = \{0\}$, by attaching $R'(x, y)$ to G_{xy}, where

$$D(x, y, R'(x, y)) = \{0\},$$

and so the resulting graph G is not C_5-colorable. If $e \in E(G')$, then there is a C_5-coloring of $G' - e$, and it can be extended to a C_5-coloring of $G - e$. Hence, $E(G')$ is contained in a C_5-critical subgraph H of G. We must have

$$|V(G')| < |V(H)| \leqslant |V(G)|, \tag{14}$$

and

$$|E(G')| < |E(H)| \leqslant |E(G)|, \tag{15}$$

and since (12) and (13) force equalities in (14) and (15), we have $H = G$. Thus, G is C_5-critical, as claimed. ∎

From (12) and (13), we get:

COROLLARY. *If G is C_5-critical and has no TK_4 subgraph, then*

$$3\,|E(G)| + 3 = 4\,|V(G)|,$$

and

$$|V(G)| \equiv 3 \pmod 9.$$

THEOREM 4. *If G is obtained from R by repeated applications of the three operations of Theorem 3, then G is R-colorable.*

Proof by Induction. R is R-colorable.

Suppose that G' has an R-coloring θ', and that G is obtained from G' by a single application of one of the three operations of Theorem 3.

Define an R-coloring θ of G by setting $\theta = \theta'$ on $G' - A_3(x, z)$ (operation 1), $G' - xy$ (operation 2), or $G' - u$ (operation 3), respectively, depending upon which operation is used to obtain G from G'. It is easy to verify that θ can be extended to the incremental subgraph that is added to G' to form G, such that θ becomes a homomorphism of G onto R. ∎

Next, we show that Theorem 2 is best-possible.

THEOREM 5. *There are infinitely many C_5-critical TK_4-free graphs with exactly two incremental subgraphs.*

Proof. Let $t \geqslant 1$, and let H be the graph consisting of the edge-disjoint incremental subgraphs $F_1 = R'(x_1, y_1)$ and $F_2 = R_0(x_t, y_t)$, and, if $t \geqslant 3$, then $2t-4$ isolated vertices $\{x_2, y_2, x_3, y_3, ..., x_{t-1}, y_{t-1}\}$. Thus, $F_1 \cap F_2 = \{x_1, y_1\}$ if $t = 1$, and F_1 and F_2 are disjoint if $t \geqslant 2$. Define G to be the graph obtained from H by the addition of these internally disjoint arcs:

$$A_2(x_{i+1}, y_{i+1}) \qquad (1 \leqslant i \leqslant t-1);$$
$$A_2(x_i, x_{i+1}) \qquad (1 \leqslant i \leqslant t-1);$$
$$A_3(x_i, x_{i+1}) \qquad (1 \leqslant i \leqslant t-1);$$
$$A_2(y_i, y_{i+1}) \qquad (1 \leqslant i \leqslant t-1);$$
$$A_3(y_i, y_{i+1}) \qquad (1 \leqslant i \leqslant t-1).$$

Thus, $|V(G)| = 9t + 12$, and the only three vertices of degree 4 in G join 5-cycles in $F_1 \cup F_2$. Since every incremental subgraph has a vertex of degree 4, F_1 and F_2 are the only incremental subgraphs in G. Since G can be obtained from R by repeated applictions of operation 1 (or 3) of Theorem 3, G is C_5-critical and K_4-free. ∎

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