Supereulerian graphs, collapsible graphs, and four-cycles

Paul A. Catlin, Department of Mathematics,
Wayne State University, Detroit, MI 48202

Abstract

Consider the problem of whether a given graph is supereulerian; i.e., whether it has a spanning eulerian subgraph. A subgraph H of G is called collapsible if for every even set $X \subseteq V(H)$, H has a spanning connected subgraph whose set of odd-degree vertices is X. For determining whether G is supereulerian, we previously showed that there is no loss of generality in contracting collapsible subgraphs to single vertices: if H is a collapsible subgraph of G, then G is supereulerian (resp., collapsible) if and only if G/H is supereulerian (resp., collapsible). This reduces the supereulerian graph problem to the case where G has no collapsible subgraph except K_1 (e.g., G is simple, G has no C_3, and the arboricity of G is at most 2). In this paper we refine this reduction method so that certain subgraphs, such as four-cycles, can be handled.

I. Introduction. We shall use Bondy and Murty's notation [4].

For an even subset S in $V(G)$, define an S-subgraph of G to be any subgraph Γ such that

(1) $$G - E(\Gamma) \text{ connected}$$

and

(2) $$S = \{v \in V(G) \mid d_\Gamma(v) \text{ is odd}\}.$$

If for every even subset $S \subseteq V(G)$, the graph G has an S-subgraph, then G is called collapsible. (This is equivalent to the definition in the abstract. The use of S-subgraphs is more convenient in the proofs.) For example, $G = C_3$ is collapsible, but $G = C_t$ $(t \geq 4)$ is not.

A graph is called supereulerian if it has a spanning eulerian subgraph. We regard the trivial graph K_1 as being both supereulerian and collapsible. Clearly, a graph G is supereulerian if and only if G has an R-subgraph, where

(3) $$R = \{v \in V(G) \mid d_G(v) \text{ is odd}\}.$$

We shall generalize the concept of collapsibility to create a reduction method (Theorem 10 and its corollaries) for showing that certain graphs are collapsible. The method can be applied to

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subgraphs such as four-cycles. Paulraja [20] has conjectured that if a graph G with $\delta(G) \geq 3$ and $\kappa(G) \geq 2$ has each of its edges in a four-cycle, then G is supereulerian. We conjecture that such a graph is also collapsible, and Theorem 10 and its corollaries are also intended as tools for proving these conjectures.

2. Previous results. Most of the results in this section are from [7].

Theorem 1 [7] Let G be a graph and let $S \subseteq V(G)$. If G has a spanning tree T such that each component of $G - E(T)$ has evenly many vertices in S, then G has an S-subgraph.

Theorem 2 Each of the following conditions implies the next:

(a) G has two edge-disjoint spanning trees;

(b) G is collapsible;

(c) G is supereulerian.

Theorems 1 and 2 are easy to prove. Jaeger [14] proved (a) implies (c), and in [7] we noted that by Theorem 1, (a) implies (b). That (b) implies (c) follows from the definitions, as we noted for (3).

Let G be a graph, let H be a subgraph of G, and let S be an even subset of $V(G)$. Define H' to be the spanning subgraph of G with $E(H) = E(H')$. We define the contraction G/H to be the graph whose vertices are the components of H', where distinct vertices of G/H are joined in G/H by as many edges as join the corresponding components of H' in G. For $e \in E(G)$, let G/e denote $G/G[e]$. We define S/H to be the subset consisting of those vertices of G/H such that the corresponding component of the spanning subgraph H' has an odd number of vertices in S.

Consider these two statements:

(4) G has an S−subgraph;

(5) G/H has an (S/H)−subgraph.

Theorem 3 [7] Let H be a connected subgraph of G, and let $S \subseteq V(G)$. Then (4) implies (5), and if H is collapsible, then (5) implies (4).

Corollary [7] If H is a collapsible subgraph of G, then G is supereulerian (resp., collapsible) if and only if G/H is supereulerian (resp., collapsible).

The supereulerian part of the corollary follows by using $S = R$ of (3) in Theorem 3. We conjecture a converse of this part:

Conjecture 1 Let H be a graph. If H is not collapsible, then H has a supergraph G such that this equivalence is false:

G is supereulerian if and only if G/H is supereulerian.

Examples: If $H = K_{2,t}$ for t even, then $G = K_{2,t+1}$ satisfies Conjecture 1. If H is not supereulerian, then $G = H$ satisfies Conjecture 1, because $G/H = K_1$ is considered supereulerian.

Theorem 4 [7] If H and H' are collapsible subgraphs of G, and if $H \cap H'$ is not empty, then $H \cup H'$ is collapsible.

Hence, there is a unique family $H_1, H_2, ..., H_c$ of maximal collapsible subgraphs of G (a collapsible subgraph is connected). Define the reduction of G to be the graph G_1 obtained from G by contracting each H_i $(1 \leq i \leq c)$ to a distinct vertex. Thus, G is supereulerian if and only if G_1 is supereulerian. Also, define G to be reduced if G is the reduction of some graph. In [7], we proved:

Theorem 5 A graph is reduced if and only if every collapsible subgraph is trivial.

Corollary If G is reduced, then every nontrivial subgraph of G is reduced.

Theorem 6 If H is a collapsible subgraph of G, then the reduction of G equals the reduction of G/H.

Let $a(G)$ denote the arboricity of G; i.e., the minimum number of edge-disjoint forests whose union equals G. Let $\omega(G)$ denote the number of components of G. We define $F(G)$ to be the minimum number of edges that must be added to G in order to obtain a spanning supergraph G' of G such that G' has two edge-disjoint spanning trees. Tutte [21] and Nash-Williams [18] proved

$$F(G) = \max_{E \subseteq E(G)} 2(\omega(G - E) - 1) - \mid E \mid .$$

In [7], we applied this formula and the arboricity formula of Nash-Williams [19], to prove:

Theorem 7 If G is reduced, then $a(G) \leq 2$; if $a(G) \leq 2$ and if G has order n, then

$$F(G) + \mid E(G) \mid = 2n - 2. \tag{6}$$

Theorem 8 [7] If $F(G) \leq 1$, then exactly one of the following holds:
(a) G is collapsible;
(b) G has a cut-edge.

Corollary If $F(G) \leq 1$ and G is reduced, then $G \in \{K_1, K_2\}$.

The case $F(G) = 0$ of Theorem 8 is $(a) \Rightarrow (b)$ of Theorem 2. If $u, v \in V(G)$ are separated by a cut-edge of G, then G has no S-subgraph when $S = \{u, v\}$, and hence G is not collapsible. The reduced graphs $G = 2K_1$ and $G = K_{2,t}$ $(t \geq 1)$ have $F(G) = 2$ and show that the bound on $F(G)$ in Theorem 8 and its corollary is best-possible.

In [7], [9], [10], and [11], we applied the previous theorems to get various conditions for a graph G to be supereulerian, thereby proving conjectures of Bauer [2] and of Benhocine, Clark, Köhler, and Veldman [3], and extending results of Brualdi and Shanny [5], Clark [13], Catlin [6], Lesniak-Foster and Williamson [17], Veldman [22], and Zhan [23]. Lai [15] has used this reduction method to prove some conjectures of Bauer [1], and in [16] Lai sharpened some results due to Chartrand and Wall [12] and Lesniak-Foster and Williamson [17].

Theorem 9 [8] If G is a collapsible graph of order n, then

$$| E(G) | \geq \frac{3}{2}(n-1). \tag{7}$$

Furthermore, for every odd natural number n, except $n = 5$, there is a collapsible graph G of order n satisfying (7) with equality and containing no nontrivial collapsible proper subgraph.

We shall prove in section 4 that the condition $\Delta(G) \leq 3$ can be added to the second part of Theorem 9. There is no collapsible subgraph of order $n = 5$ that does not contain the collapsible subgraph C_3.

Conjecture 2 Let $G(e)$ denote the graph obtained from G with an elementary subdivision of $e \in E(G)$. If G is a collapsible graph, then either G satisfies (7) with equality, or there is an edge $e \in E(G)$ such that $G(e)$ is collapsible.

3. **A generalization of collapsibility.** Let H be a graph and let π be a partition of $V(H)$ into two sets V_1 and V_2. Then H is called π^--collapsible (resp., π^+-collapsible) if for every even set $R \subseteq V(H)$,

(i) If $| R \cap V_1 |$ is odd (resp., even), then H has an R-subgraph Γ;

(ii) If $| R \cap V_1 |$ is even, (resp., odd), then $H + e$ has an R-subgraph $\Gamma_e \subseteq H$, for any added edge $e = w_1 w_2$ with $w_1 \in V_1$ and $w_2 \in V_2$.

If H is either π^--collapsible or π^+-collapsible, then H is called π-collapsible.

A collapsible graph H is π^--collapsible and π^+-collapsible for any 2-partition π of $V(H)$.

Example 1 Suppose that H is the 4-cycle $abcda$, and let $V_1 = \{a, c\}$, $V_2 = \{b, d\}$ be the 2-partition π of $V(H)$. We show that H is π^--collapsible. Let R be an even subset of $V(H)$. If $| R \cap V_1 |$ is odd, then H has an R-subgraph Γ, with $E(\Gamma)$ consisting of a single edge whose ends are the two vertices of R. Suppose $| R \cap V_1 |$ is even. If $R = V(H)$, then there is a subgraph Γ_e satisfying (ii), where $E(\Gamma_e)$ is a perfect matching of H, such that one edge of

$E(\Gamma_e)$ is parallel to e. If R is nonempty, then by symmetry, all other cases are equivalent to $R = \{a, c\}$ and $e = ab$ in (ii). In this case, when Γ_e satisfies $E(\Gamma_e) = \{ab, bc\}$, (ii) holds. If R is empty, then (ii) holds with Γ_e edgeless, regardless of e.

Example 2 The TK_4 of order 8 and girth 5 is π^--collapsible, if π partitions the vertex set into divalent and trivalent vertices. We prove this in Theorem 12. (A TK_4 is any graph obtained from K_4 by subdivisions.)

Example 3 Let G be the bipartite graph obtained from the 6-cycle $abcdefa$ by the addition of the edge ad. For the partition $\pi = \{a, c, e\} \cup \{b, d, f\}$, the graph G is π^+-collapsible, as we shall show, following the corollaries of Theorem 10.

Example 4 Suppose that a connected graph G has a cut-edge e, and suppose that the components of $G - e$ are each collapsible. Then G is π^+-collapsible, where each equivalence class of π is the vertex set of a component of $G - e$.

Example 5 If x and y are nonadjacent vertices of degree 2 in $G = K_{2,t}$, where $t \geq 2$, then G is π^--collapsible, where $\{x, y\}$ and $V(G) - \{x, y\}$ are the equivalence classes of π. We prove this following the corollaries of Theorem 10.

Conjecture 3 Let G be a graph. If H is a K_2 in G such that G/H is collapsible, then G is π-collapsible, for some 2-partition π of $V(G)$.

Conjecture 4 If a collapsible graph G has a vertex v of degree 2, then $G - v$ is π-collapsible for some 2-partition π of $V(G - v)$.

Notation: To avoid subscripts on subscripts, we shall use $deg_\pi(v)$, $deg_0(v)$, and $deg_R(v)$, respectively, to denote the degree of v in Γ_π, Γ_0, and Γ_R, respectively. The graphs Γ_π, Γ_0, and Γ_R will be defined in what follows. The degree of v in Γ will be denoted $d_\Gamma(v)$, and $d(v)$ will denote the degree of v in G.

Let H be a π-collapsible subgraph of G, where V_1 and V_2 are the equivalence classes of π in $V(H)$. Denote by G/π the graph obtained from G by identifying all vertices of V_1 to form a single vertex v_1, by identifying all vertices of V_2 to form a single vertex v_2, and by joining v_1 and v_2 with exactly one edge. Thus, each v_i $(i = 1, 2)$, is joined in $(G/\pi) - v_1v_2$ to any $w \in V(G) - V_1 - V_2$ by as many edges as join w and V_i in G. For $S \subseteq V(G)$,, define S/π to include $S - V_1 - V_2$ and every v_i $(i = 1, 2)$ such that $| S \cap V_i |$ is odd.

Theorem 10 Let G be a graph, $S \subseteq V(G)$, and H be a π-collapsible induced subgraph of G, where π partitions $V(H)$ into sets V_1 and V_2 corresponding to v_1 and v_2, respectively, in $V(G/\pi)$. If both

(8) $$H \text{ is } \pi^- \text{-collapsible}$$

and

(9) $$G/\pi \text{ has an } ((S/\pi)\Delta\{v_1, v_2\})\text{-subgraph},$$

or if both

(10) $$H \text{ is } \pi^+\text{-collapsible}$$

and

(11) $$G/\pi \text{ has an } (S/\pi) - \text{subgraph},$$

then

(12) $$G \text{ has an } S\text{-subgraph}.$$

Proof: Let G, S, H, and π satisfy the hypothesis, and let v_i $(i = 1, 2)$ be the vertex of G/π corresponding to the equivalence class V_i.

Suppose that both (8) and (9) or both (10) and (11) hold. Then either

(13) $$\text{(8) holds and } G/\pi \text{ has an } ((S/\pi)\Delta\{v_1, v_2\})\text{-subgraph } \Gamma_\pi$$

or

(14) $$\text{(10) holds and } G/\pi \text{ has an } (S/\pi)\text{-subgraph } \Gamma_\pi.$$

By (13), (14), (1), and (2),

(15) $$G/\pi - E(\Gamma_\pi) \text{ is connected},$$

(16) $$\text{(8) implies } (S/\pi)\Delta\{v_1, v_2\} = \{v \in V(G/\pi) \mid deg_\pi(v) \text{ is odd}\},$$

(17) $$\text{(10) implies } (S/\pi) = \{v \in V(G/\pi) \mid deg_\pi(v) \text{ is odd}\}.$$

Since $E(\Gamma_\pi - v_1v_2) \subseteq E(G/\pi - v_1v_2) \subseteq E(G)$, we can define

(18) $$\Gamma_0 = G[E(\Gamma_\pi - v_1v_2)],$$

(19) $$S_0 = \{v \in V(G) \mid deg_0(v) \text{ is odd}\},$$

and note that since Γ_0 has an even number of odd-degree vertices,

(20) $$\mid S_0 \mid \text{ is even.}$$

Let $e \in E(G) - E(H)$. Then $e \in E(\Gamma_0)$ iff $e \in E(\Gamma_\pi)$, and so

(21) $$S_0 - V(H) = S - V(H).$$

It follows from (16) and (17) that $|(S/\pi)\Delta\{v_1, v_2\}|$ and $|S/\pi|$ are even, and hence $|S|$ is even. This and (20) imply that the set

(22) $$R = S\Delta S_0$$

has even cardinality, and (21) and (22) imply

$$R \subseteq V(H). \tag{23}$$

Suppose that H has an R-subgraph Γ_R, and define

$$\Gamma = \Gamma_R \cup \Gamma_0. \tag{24}$$

Then $H_0 = H - E(\Gamma_R)$ is connected. Let $H_1, H_2, ..., H_k$ denote the components of $G - E(\Gamma_0) - E(H)$. By (15), (18), and the definition of G/π, each H_i $(1 \leq i \leq k)$ has a vertex in $V_1 \cup V_2 = V(H)$. Since $E(\Gamma_R) \subseteq E(H)$ and $E(\Gamma_0) \subseteq E(G) - E(H)$, $E(\Gamma_R)$ and $E(\Gamma_0)$ are disjoint. Hence, (24) implies

$$\begin{aligned} G - E(\Gamma) &= G - E(\Gamma_0) - E(\Gamma_R) \\ &= G - E(\Gamma_0) - (E(H) - E(H_0)) \\ &= H_0 \cup H_1 \cup ... \cup H_k, \end{aligned}$$

and since each H_i $(0 \leq i \leq k)$ is connected and intersects H_0,

$$G - E(\Gamma) \text{ is connected.} \tag{25}$$

Since Γ_R is an R-subgraph of H,

$$R = \{v \in V(H) \mid deg_R(v) \text{ is odd}\},$$

which, along with (22), (19), (23), and (24), implies

$$\begin{aligned} S &= S_0 \Delta R \\ &= \{v \in V(G) \mid deg_0(v) \text{ is odd}\} \Delta \{v \in V(H) \mid deg_R(v) \text{ is odd}\} \\ &= \{v \in V(G) \mid d_\Gamma(v) \text{ is odd}\}. \end{aligned} \tag{26}$$

By (26) and (25), Γ is an S-subgraph of G, and so (12) holds.

Hence, we may suppose that H has no R-subgraph. Since H is π-collapsible, (ii) of the definition holds, and so

$$\text{(8) implies} \quad | R \cap V_1 | \text{ is even} \tag{27}$$

and

$$\text{(10) implies} \quad | R \cap V_1 | \text{ is odd.} \tag{28}$$

<u>Case 1</u> Suppose $v_1 v_2 \notin E(\Gamma_\pi)$. Then by (18),

$$\Gamma_0 = G[E(\Gamma_\pi)],$$

and so by (19) and since $v_1 \in V(G/\pi)$ corresponds to $V_1 \subseteq V(G)$,

$$deg_\pi(v_1) = deg_0(v_1) \equiv | S_0 \cap V_1 | \pmod 2. \tag{29}$$

By the definition of S/π,

$$(30) \qquad | S \cap V_1 | \equiv | (S/\pi) \cap \{v_1\} | \pmod 2.$$

By (16) and (17),

$$(31) \qquad | (S/\pi) \cap \{v_1\} | \begin{cases} \equiv & deg_\pi(v_1) \text{ if (10) holds;} \\ \not\equiv & deg_\pi(v_1) \text{ if (8) holds.} \end{cases}$$

By (29) and (22),

$$(32) \qquad \begin{aligned} deg_\pi(v_1) &\equiv | S_0 \cap V_1 | \\ &\equiv | (R \Delta S) \cap V_1 | \\ &\equiv | (R \cap V_1) \Delta (S \cap V_1) | . \end{aligned}$$

By (27) and (28),

$$(33) \qquad | (R \cap V_1) \Delta (S \cap V_1) | \begin{cases} \not\equiv & | S \cap V_1 | \text{ if (10) holds.} \\ \equiv & | S \cap V_1 | \text{ if (8) holds.} \end{cases}$$

By (30), (31), (32), and (33), we have

$$(34) \qquad | S \cap V_1 | \not\equiv | S \cap V_1 | \pmod 2,$$

a contradiction. Hence, Case 1 is impossible.

<u>Case 2</u> Suppose

$$v_1 v_2 \in E(\Gamma_\pi).$$

By (15), $G/\pi - E(\Gamma_\pi)$ is connected and contains a (v_1, v_2)-path P_π, and by the condition of Case 2, $v_1 v_2 \notin E(P_\pi)$. Hence, $G - E(\Gamma_0) - E(H)$ has a path

$$P = G[E(P_\pi)]$$

whose ends we denote $x_1 \in V_1$ and $x_2 \in V_2$. Since H is π-collapsible, $H + x_1 x_2$ has an R-subgraph Γ_R, and so the subgraph

$$H_0 = (H \cup P) - E(\Gamma_R)$$

is a connected subgraph of $G - E(\Gamma_0) - E(\Gamma_R)$. Set

$$(35) \qquad \Gamma = \Gamma_0 \cup \Gamma_R.$$

Let $H_1, H_2, ..., H_k$ denote the components of $G - E(\Gamma_0) - E(H)$. By (15), (18), and the definition of G/π, each H_i $(1 \le i \le k)$ has a vertex in $V_1 \cup V_2 \subseteq V(H_0)$. Hence,

$$\begin{aligned} G - E(\Gamma) &= G - E(\Gamma_0) - E(\Gamma_R) \\ &= G - E(\Gamma_0) - (E(H) - E(H_0)) \\ &= H_0 \cup H_1 \cup ... \cup H_k, \end{aligned}$$

and since each H_i $(0 \leq i \leq k)$ is connected and intersects H_0,

(36) $$G - E(\Gamma) \text{ is connected.}$$

Now, by (22), (19), the definition of Γ_R, (23), (35), and since $E(\Gamma_0) \cap E(\Gamma_R)$ is empty,

(37) $$\begin{aligned} S &= S_0 \Delta R \\ &= \{v \in V(G) \mid deg_0(v) \text{ is odd}\} \Delta \{v \in V(H) \mid deg_R(v) \text{ is odd}\} \\ &= \{v \in V(G) \mid d_\Gamma(v) \text{ is odd}\}. \end{aligned}$$

By (36) and (37), Γ satisfies (1) and (2), whence Γ is an S-subgraph of G, and so (12) holds. This completes the proof. □

Corollary 1 If H is a π-collapsible subgraph of G for some 2-partition π of $V(H)$, and if G/π is collapsible, then G is collapsible.

Proof: Let H be a π-collapsible subgraph of G, and suppose that G/π is collapsible. Let S be an even subset of $V(G)$. Then $(S/\pi)\Delta\{v_1, v_2\}$ and S/π are even subsets of $V(G/\pi)$, and since G/π is collapsible, (9) and (11) of Theorem 10 hold. Since H is π- collapsible, either (8) or (10) holds. By Theorem 10, G has an S-subgraph. Since S is arbitrary, G is collapsible. □

Corollary 2 Let H be a subgraph of G, and let π be a 2-partition of H. If H is π-collapsible and if

$$G/\pi \text{ is supereulerian,}$$

then

$$G \text{ is supereulerian.}$$

Proof: Define

(38) $$S = \{v \in V(G) \mid d(v) \text{ is odd}\}.$$

Either

(39) $$S/\pi = \{v \in V(G/\pi) \mid d_{G/\pi}(v) \text{ is odd}\}$$

or for v_1 and v_2 defined as in Theorem 10,

(40) $$(S/\pi)\Delta\{v_1, v_2\} = \{v \in V(G/\pi) \mid d_{G/\pi}(v) \text{ is odd}\}$$

Suppose that G/π has a spanning eulerian subgraph, say G_0. Since G_0 is 2-edge-connected, the induced graph $(G/\pi)[E(G_0)\Delta\{v_1v_2\}]$ is connected, and hence consists of a spanning (v_1, v_2)-trail of G. Therefore, G/π has both an (S/π)-subgraph and an $((S/\pi)\Delta\{v_1, v_2\})$-subgraph, where S/π satisfies (39) and $(S/\pi)\Delta\{v_1, v_2\}$ satisfies (40). Hence, both (9) and (11) of Theorem 10 hold. Since H is π-collapsible, either (8) or (10) holds. By Theorem 10, G has an S-subgraph Γ, and since S satisfies (38), $G - E(\Gamma)$ is a spanning eulerian subgraph of G. □

Definitions: Let π be the 2-partition $V(G) = X_1 \cup X_2$, and let H be a subgraph of G. The restriction of π to H, denoted π_H, is the 2-partition $V(H) = V_1 \cup V_2$, where

$$V_1 = X_1 \cap V(H), \quad V_2 = X_2 \cap V(H).$$

The contraction of π to G/π_H, denoted π/π_H, is the 2-partition

$$V(G/\pi_H) = (X_1/V_1) \cup (X_2/V_2).$$

Corollary 3 Let H be a subgraph of G, and let π be a 2-partition of $V(G)$. If

(41) H is π_H^+–collapsible and G/π_H is $(\pi/\pi_H)^+$–collapsible

or if

(42) H is π_H^-–collapsible and G/π_H is $(\pi/\pi_H)^-$–collapsible,

then G is π^+-collapsible. If instead,

(43) H is π_H^+–collapsible and G/π_H is $(\pi/\pi_H)^-$–collapsible

or if

(44) H is π_H^-–collapsible and G/π_H is $(\pi/\pi_H)^+$–collapsible,

then G is π^--collapsible.

Proof: Let H be a subgraph of G, and let π be a 2-partition of $V(G)$ into sets X_1 and X_2. Define

$$V_1 = X_1 \cap V(H); \quad V_2 = X_2 \cap V(H),$$

and let v_1 (resp., v_2) be the vertex of G/π_H corresponding to V_1 (resp., V_2).

Case 1: Suppose (41) holds. Let S be an even subset of $V(G)$. Then S/π_H is an even subset of $V(G/\pi_H)$. By (41), G/π_H is $(\pi/\pi_H)^+$-collapsible, and so we have

(45) G/π_H has an (S/π_H)–subgraph

or both

(46) $|\,(S/\pi_H) \cap (X_1/V_1)\,|$ is odd

and

(47) $G/\pi_H + e$ has an (S/π_H)–subgraph Γ_e

for any added edge $e = w_1w_2$ with

(48) $$w_1 \in X_1/V_1, \quad w_2 \in X_2/V_2.$$

If (45) holds, then we have (11), and since (41) gives (10), Theorem 10 implies that

(49) G has an S–subgraph.

Suppose that (46) and (47) hold, and let $x_1 \in X_1$, $x_2 \in X_2$. Choose $e = w_1w_2$ in (47) and (48) such that $w_i = x_i$ if $x_i \notin V_i$ and $w_i = v_i$ if $x_i \in V_i$, for $i = 1$ and $i = 2$. Since $G/\pi_H + e = (G + x_1x_2)/\pi_H$, (47) implies

(50) $$(G + x_1x_2)/\pi_H \text{ has an } (S/\pi_H)\text{–subgraph } \Gamma_e.$$

By (41) we have (10), and by (50) we have (11) for $G + x_1x_2$, and so by Theorem 10,

(51) $$G + x_1x_2 \text{ has an } S\text{–subgraph.} \quad (x_1 \in X_1, x_2 \in X_2).$$

Since we have either (49) or both (46) and (51) for any even subset $S \subseteq V(G)$, G is π^+-collapsible.

Cases 2,3, and 4: Suppose (42), (43), or (44) holds. The argument of Case 1 works. □

Let G and π be the graph and partition of Example 3, and let H be the 4-cycle $abcda$ in G. By Example 1, H is π_H^--collapsible, and since G/π_H is a 4-cycle, it is also $(\pi/\pi_H)^-$-collapsible, by Example 1. By (42) of Corollary 3, G is π^+-collapsible.

Let $G = K_{2,t}$ and π be the graph and partition of Example 5, and let H be the 4-cycle containing x and y. By Example 1, H is π_H^--collapsible, and since G/π_H consists of a cut-edge v_1v_2 separating two collapsible subgraphs (the components of $G/\pi_H - v_1v_2$), G/π_H is $(\pi/\pi_H)^+$-collapsible, by Example 4. By (44) of Corollary 3, G is π^--collapsible.

Lemma 1 If G is a TK_4 of order at most 6, then G is collapsible.

Proof: Let G be a TK_4 of order at most 6. If G has a triangle H, then H and G/H are collapsible, and so G is collapsible, by the Corollary of Theorem 3.

Suppose G is triangle-free. Then the hypothesis of the lemma implies that G has order 6 and has a 4-cycle, say H, that contains a vertex whose degree in G is 2. By Example 1, H is π-collapsible, where π is a proper 2-coloring of H. Then

$$\kappa'(G/\pi) \geq 2, \qquad a(G/\pi) \leq 2,$$

and so by Theorem 7,

$$F(G/\pi) = 2\,|V(G/\pi)| - 2 - |E(G/\pi)| = 2(4) - 2 - 5 = 1.$$

Hence, by Theorem 8, G/π is collapsible, and so by Corollary 1 G is collapsible. □

Let G be the unique triangle-free TK_4 of order 6, and let H be the four-cycle induced by vertices of degree 3 in G. Let the partition π be a proper 2-coloring of H. Then H is π-collapsible and since G/π has a cut-edge, G/π is not collapsible. But, by Lemma 1, G is collapsible, and so the converse of Corollary 1 of Theorem 10 is false.

4. Collapsible graphs with low density. The graphs of the next theorem satisfy (7) of Theorem 9 with equality.

Theorem 11 Let G be the graph of order $2t+3$ with $3t+3$ edges, with

$$V(G) = \{x_0, x_1, ..., x_t, y_0, y_1, ..., y_t, v\},$$

where $E(G)$ consists of the edges x_0y_0, x_0v, exactly one of $\{x_tv, y_tv\}$, and (if $t > 0$) the edge set

$$\bigcup_{i=1}^{t} \{x_iy_i, x_ix_{i-1}, y_iy_{i-1}\}.$$

Then G is collapsible if and only if G has an odd cycle.

Proof: Suppose that G has no odd cycle, and let

$$S = \{w \in V(G) \mid d(w) = 3\}.$$

If Γ is an S-subgraph of G, then $G - E(\Gamma)$ must be connected and 2-regular, and hence a hamiltonian cycle. But G has odd order and no odd cycle, and so G cannot be hamiltonian. Therefore, G has no S-subgraph and cannot be collapsible.

Next, suppose that G has an odd cycle C, and suppose, inductively, that Theorem 11 holds for graphs smaller than G. Since Theorem 11 holds when $t = 0$, we have a basis for induction. Let H be the induced four-cycle $x_ix_{i-1}y_{i-1}y_ix_i$ in G, where $1 \leq i \leq t$ and i is chosen so that $E(C) \cap E(H) \neq \emptyset$. Let the partition π of $V(H)$ be a proper 2-coloring of H. Then C induces an odd cycle C' in G/π, where $E(C')$ consists of $E(C) - E(H)$ and possibly the edge v_1v_2 of the definition of the term π-collapsible, depending on the parity of $\mid E(C) \cap E(H) \mid$. By the induction hypothesis, G/π is collapsible, and since H is π-collapsible, Corollary 1 of Theorem 10 implies that G is collapsible. □

Theorem 12 Let G be a TK_4 of order 8 and girth 5. Define

$$V(G) = V_1 \cup V_2, \quad V_1 = \{a, b, c, d\}, \quad V_2 = \{u, v, w, x\},$$

and define $E(G)$ to consist of ab, cd and the edges of the hamiltonian cycle $aucwbxdva$. Then G is π^--collapsible, where π partitions $V(G)$ into sets V_1 and V_2.

Proof: First, we must show that for any even $R \subseteq V(G)$, with $\mid R \cap V_1 \mid$ odd, G has an R-subgraph. This is easily verified for $\mid R \mid \leq 2$. The following table gives the edge set of an R-subgraph Γ, for each of the sets R listed:

R	$E(\Gamma)$
$\{a,b,c,u\}$	$\{ab,cu\}$
$\{a,b,c,v\}$	$\{ab,cd,dv\}$
$\{a,u,v,w\}$	$\{av,cd,dv\}$
$\{b,u,v,w\}$	$\{au,av,bw\}$
$\{a,b,c,u,v,w\}$	$\{av,cu,bw\}$
$\{a,b,c,u,v,x\}$	$\{av,cu,bx\}$

All other even sets $R \subseteq V(G)$ with $|R|$ even and with $|R \cap V_1|$ odd are equivalent by symmetry to a set listed in the table. Hence, G satisfies (i) of the definition of π-collapsible graphs.

Next, for $|R|$ and $|R \cap V_1|$ even, we must consider graphs $G + z_1z_2$, where $z_1 \in V_1$ and $z_2 \in V_2$. We show that $G + z_1z_2$ has an R-subgraph by proving that $G + z_1z_2$ is collapsible. By symmetry, all possible values of z_1z_2 are equivalent either to $z_1z_2 = bu$ or $z_1z_2 = bw$. Suppose $z_1z_2 = bu$. With repeated applications of the corollary of Theorem 3, it is easy to show that $G + z_1z_2$ is collapsible: it has a 3-cycle $H = baub$, and $(G + z_1z_2)/H$ has a 3-cycle, etc., and 3-cycles are collapsible. Finally, suppose $z_1z_2 = bw$. Let H be the 2-cycle of $G + z_1z_2$ that contains b and w. The graph $(G + z_1z_2)/H$ is the graph G of Theorem 11 with order $2t + 3 = 7$ and $N(v) = \{x_0, y_2\}$ (v has the same meaning in $(G + z_1z_2)/H$ as it has in Theorem 11), and so $(G + z_1z_2)/H$ is collapsible, by Theorem 11. Since H is collapsible, too, the corollary of Theorem 3 implies that $G + z_1z_2$ is collapsible. □

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