Spanning Trails

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ABSTRACT

For a graph G with distinguished vertices u and v, we give a sufficient condition for the existence of a (u, v)-trail containing every vertex of G.

In this paper we follow the notation of Bondy and Murty [1], except that the graph G is simple with n vertices and m edges. For u, $v \in V(G)$, a (u, v)*-trail* is a sequence $x_0, e_1, x_1, e_2, \ldots, x_{s-1}, e_s, x_s$ whose terms are alternately vertices and edges, with e_i joining x_{i-1} and x_i $(1 \leq i \leq s)$, where the edges are distinct, and where $u = x_0$ is the *origin* and $v = x_s$ is the *terminus*. A (u,v)-trail *spans* G if it contains every vertex of G, and it is *closed* if $u = v$. We denote by $d(v)$ the degree of v in G and by $d_H(v)$ the degree of v in the subgraph H. The *neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v.

We shall prove the following result:

Theorem 1. Let G be a graph on n vertices, with no vertex isolated, and let $u, v \in V(G)$. If

$$d(x) + d(y) \geq n \tag{1}$$

for each edge $xy \in E(G)$, then exactly one of the following holds:

(i) G has a spanning (u, v)-trail.
(ii) $d(z) = 1$ for some vertex $z \notin \{u, v\}$.
(iii) $G = K_{2,n-2}$, $u = v$, and n is odd.
(iv) $G = K_{2,n-2}$, $u \neq v$, $uv \notin E(G)$, n is even, and $d(u) = d(v) = n - 2$.
(v) $u = v$, and u is the only vertex with degree 1 in G.

Theorem 1 is motivated by some recent results on Hamiltonian line graphs. Harary and Nash-Williams [5] gave this characterization:

Theorem 2 (Harary and Nash-Williams). Let G be a graph with at least 4 vertices. The line graph $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one vertex of each edge of G.

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Note that the closed trail does not need to be spanning in Theorem 2.

Of course, G has a spanning closed trail if and only if G has a spanning eulerian subgraph. Harary and Nash-Williams ([5], p. 705) also gave another characterization of graphs with spanning closed trials. Given a graph G with m edges, let $L_2(G)$ denote the graph on $2m$ vertices, where each vertex of $L_2(G)$ represents an edge-vertex incidence of G, and x, y, $\in V(L_2(G))$ are adjacent whenever x and y are incidences with a common edge or a common vertex of G.

Theorem 3 (Harary and Nash-Williams). The graph G has a spanning closed trail if and only if $L_2(G)$ is Hamiltonian.

Similarly, G has a spanning open trail if and only if $L_2(G)$ has a hamilton path.

Theorem 2 was recently applied to prove these results:

Theorem 4 (Brualdi and Shanny [2]). Let G be a graph with $n \geq 4$ vertices. If

$$d(x) + d(y) \geq n$$

for every edge $xy \in E(G)$, then $L(G)$ is Hamiltonian.

Theorem 5 (Clark [3]). Let G be a connected graph on $n \geq 6$ vertices, and let $p(n) = 0$ for n even and $p(n) = 1$ for n odd. If

$$d(x) + d(y) \geq n - 1 - p(n)$$

for each edge $xy \in E(G)$, then $L(G)$ is Hamiltonian.

It is evident that Theorem 4 follows from Theorems 1 and 2, because $L(K_{2,n-2})$ is Hamiltonian, and vertices of degree 1 can be removed from G inductively until condition (ii) does not apply.

If we replace the inequality (1) in Theorem 1 by

$$d(x) + d(y) \geq n - 1,$$

then exceptional cases would arise. One special exceptional case would be the five cycle:

$$G = C_5, \qquad uv \notin E(G), \qquad u \neq v.$$

The others include the following infinite class:

$G - u = K_{2,n-3}$, $d(u) = 1, d_{G-u}(v) = n - 3$, where $u \neq v, n \geq 5$, and either $uv \in E(G)$ with n even, or the distance in G between u and v is 3, with n odd.

The extremal graphs for Theorem 5 have a bridge e such that each component of $G - e$ has at least $[n/2]$ vertices.

The graphs $G = K_{2,n-2}$ of Theorem 1 arise in another context, also: if any edge of $K_{2,n-2}$ is removed, then the connectivity drops from 2 to 1. Among graphs on n vertices, no others having this minimality property have $2n - 4$ or more edges. See [4] and [6] for details.

Proof of Theorem 1. It is clear that the conditions (i) through (v) of the theorem are mutually exclusive.

Suppose that G is the smallest counterexample to the theorem. Let $u, v \in V(G)$ be given.

Let $\gamma_{u,v}$ be a (u, v)-trail of G that has the maximum possible number of vertices, excluding multiplicities. Since G has no isolated vertex, we can show from (1) that G is connected. Therefore, $\gamma_{u,v}$ exists. Let A be the vertex set of $\gamma_{u,v}$. Let $B = V(G) - A$, and denote $H = G[B]$.

We shall use the following lemmas:

Lemma 1. There is no closed trail μ in G containing a vertex of A, a vertex of B, but at most one edge of $\gamma_{u,v}$.

This result is given in [2, (1) and (2), p. 308], when μ is a cycle, as a simple consequence of the fact that a trail is formed by the union of a trail γ and a cycle μ that overlaps γ in at least one vertex and at most one edge e, where e is not in the enlarged trail. When μ is a closed trial, it is an edge-disjoint union of cycles, and so the lemma holds. ∎

Let Γ denote the subgraph of $G[A]$ that is induced by the edges of $\gamma_{u,v}$.

Let H' be a component of $H = G[B]$. Define $N(H') = \{w \in A \mid wx \in E(G) \text{ for some } x \in V(H')\}$.

Lemma 2. Let $z \in A$, and suppose

$$y_1, y_2 \in N(H') \cap N(z).$$

Then $y_1 z$, $y_2 z \in E(\Gamma)$.

Proof. Let y_1, y_2 be as described in the lemma. Suppose that at most one of $y_1 z, y_2 z$ lies in Γ. For $i = 1, 2$, choose $x_i \in N(y_i) \cap V(H')$. Let μ be the cycle containing an (x_1, x_2)-path in H' and edges of $\{x_1 y_1, y_1 z, z y_2, y_2 x_2\}$. By Lemma 1, with μ and $\gamma_{u,v}$, we have a contradiction. Thus, both $y_1 z$ and $y_2 z$ are in Γ. ∎

Lemma 3. Let A and H' be as previously defined. If $z \in A$ then

$$|N(z) \cap N(H')| \leq 2.$$

Proof. Suppose, by way of contradiction, that $y_1, y_2, y_3 \in N(z) \cap N(H')$. By Lemma 2, $y_1 z, y_2 z, y_3 z \in E(\Gamma)$.

Since y_i, y_j $(1 \leq i < j \leq 3)$ are both adjacent to vertices of the same component H' of H, there is a (y_i, y_j)-path γ_{ij}, with a nonempty set X_{ij} of internal vertices in H'.

We shall use the fact (Euler's theorem) that a graph has a (u, v)-trail using every edge of the graph if and only if the graph is connected and each vertex has even degree, except that disjoint endvertices u and v have odd degree.

Let $1 \leq i < j \leq 3$. Since $\gamma_{u,v}$ contains each edge of Γ exactly once, each vertex of Γ has even degree, except that u and v have odd degree in Γ if $u \neq v$. Therefore, every vertex of

$$\Gamma_{ij} = (\Gamma \cup \gamma_{ij}) - \{zy_i, zy_j\}$$

has even degree, except for u and v if $u \neq v$.

Thus, $V(\Gamma_{ij}) = A \cup X_{ij}$. If Γ_{ij} is connected, then its spanning (u, v)-trail violates the maximality of A. Hence, Γ_{ij} is not connected, and so the removal of $\{zy_i, zy_j\}$ must separate Γ, for any choice of i, j.

First, suppose that none of $\{zy_1, zy_2, zy_3\}$ is a bridge of Γ. Then in $\Gamma - zy_1$, both zy_2 and zy_3 are bridges. Denote by Γ_2 and Γ_3, respectively, the components of $\Gamma - \{zy_1, zy_2, zy_3\}$ that contain, respectively, y_2 and y_3. For some value of $i \in \{2, 3\}$, $y_1 \notin V(\Gamma_i)$. Thus, zy_i is a bridge of Γ, contrary to our earlier assumption.

Therefore, without loss of generality, we suppose that zy_1 is a bridge of Γ. Since $\gamma_{u,v}$ contains each edge of Γ exactly once, u and v are in separate components of $\Gamma - zy_1$. Denote by Γ_u and Γ_v the two components of $\Gamma - zy_1$, where $u \in V(\Gamma_u)$ and $v \in V(\Gamma_v)$.

Case 1. Suppose $uv \in E(G)$. Then $uv = zy_1$, and without loss of generality, we suppose that $u = y_1$ and $v = z$. Therefore, $y_2, y_3 \in V(\Gamma_v)$. Observe that Γ_u and Γ_v are each eulerian.

Pick $j \in \{2, 3\}$. Clearly, $v \neq y_j$. Since Γ_v is eulerian, $\Gamma_v - zy_j$ has a (z, y_j)-trail γ_v using every edge just once. Let γ_u be the eulerian trail in Γ_u. Then $\gamma_u \cup \gamma_{ij} \cup \gamma_v$ forms a spanning (u, v)-trail of $G[A \cup X_{ij}]$, contrary to the maximality of A.

Case 2. Suppose that $uv \notin E(G)$. Thus, $uv \neq zy_1$. Without loss of generality, suppose $y_1 \in V(\Gamma_u)$ and $z \in V(\Gamma_v)$. As sections of the (u, v)-trail $\gamma_{u,v}$ forming Γ, we have a (u, y_1)-trail γ_u containing $E(\Gamma_u)$ and a (z, v)-trail γ_v containing $E(\Gamma_v)$.

Since $zy_2, zy_3 \in E(\Gamma_v)$, either

(a) $z = v$ and v has even degree in Γ_v, or
(b) $z \neq v$ and z has odd degree, at least 3, in Γ_v.

In case (b), since $d(z) \geq 3$ in Γ_v, and since γ_v is a (z, v)-trail on all of $E(\Gamma_v)$, there is a number $j \in \{2, 3\}$ such that $v \neq y_j$ and zy_j is not a bridge of Γ_v. Then

$\Gamma_v - zy_j$ is connected, with exactly two odd vertices (v and y_j), and so $\Gamma_v - zy_j$ has a (y_j, v)-trail γ'_v using each edge. A (u, v)-trail on all of $A \cup X_{lj}$ is formed by γ_u, γ_{lj}, and γ'_v together, contrary to the maximality of A.

Similarly, in case (a), when $z = v$ has even degree in Γ_v, there is a $j \in \{2, 3\}$ such that $v \neq y_j$ and $\Gamma_v - zy_j$ has a (y_j, v)-trail γ'_v containing every edge. The same combination as before of $\gamma_u, \gamma_{lj}, \gamma'_v$ contradicts the maximality of A. This completes the proof of Lemma 3. ∎

Lemma 4. If there is a subset $X \subseteq V(G)$ such that $G[X]$ contains an edge, and such that a bridge of G separates $G[X]$ from $G - X$, then

$$|X| \geq \frac{n+1}{2}.$$

Proof. Let $xy \in E(G[X])$. Let $d_X(z)$ denote the degree of z in $G[X]$. By the hypothesis of Lemma 4 and by (1),

$$d_X(x) + d_X(y) \geq d(x) + d(y) - 1 \geq n - 1.$$

Without loss of generality, suppose $d_X(x) \geq d_X(y)$. Then

$$|X| \geq 1 + d_X(x) \geq \frac{n+1}{2}. \quad ∎$$

Proof of Theorem 1 Continued. Let H' be a component of $H = G[B]$, where H' is chosen to maximize the number s of vertices of

$$N(H') = \{y_1, y_2, \ldots, y_s\}.$$

We need an upper bound on k, the number of edges incident with vertices of $N(H')$. Since $N(H')$ is an independent set (Lemma 1), we have

$$k = \sum_{i=1}^{s} |N(y_i) \cap (A - N(H'))| + \sum_{i=1}^{s} |N(y_i) \cap B|. \tag{2}$$

By Lemma 3, the first sum of (2) is bounded above by

$$2(|A| - |N(H')|) = 2(|A| - s),$$

and by Lemma 1 and the choice of H', the second sum of (2) is bounded above by cs, where c is the number of components of H. Hence,

$$k \leq 2(|A| - s) + cs,$$

and some $y_i \in N(H')$ $(1 \leq i \leq s)$ must satisfy

$$d(y_i) \leq \frac{k}{s} \leq \frac{2|A|}{s} - 2 + c\,.$$

By (1), it follows that

$$n \leq d(x) + d(y_i) \leq \frac{2|A|}{s} - 2 + c + s\,, \tag{3}$$

for any $x \in N(y_i) \cap B$. Therefore,

$$sn \leq 2|A| + s^2 + sc - 2s \leq 2(n - c) + s^2 + sc - 2s$$
$$n(s - 2) \leq s^2 + sc - 2c - 2s = (s - 2)(s + c)\,,$$

and so, if $s \geq 3$, then we divide both sides by $s - 2$ and get

$$|A| + |B| = n \leq s + c \leq |A| + c\,. \tag{4}$$

By the definition of c, $|B| = c$ follows, for equality holds in (4). Thus, $s = |A|$, and so $N(H') = A$, contrary to the maximality of A, unless $|A| = 1$. If $|A| = 1$, then (v) of Theorem 1 holds.

Therefore, $s \leq 2$, and we may suppose $|A| \geq 2$. Thus, $G[A]$ has an edge. If $s = 1$, then $G - H'$ and H' are joined by a bridge, by Lemma 1. By Lemma 4, $|V(G - H')| \geq (n + 1)/2$. If $|V(H')| = 1$, then $V(H') = \{z\}$ satisfies (ii) of Theorem 1. If $|V(H')| \geq 2$, then by Lemma 4, $|V(H')| \geq (n + 1)/2$, and so

$$n = |V(H')| + |V(G - H')| \geq n + 1\,,$$

a contradiction. Hence, $s = 2$.

Plug $s = 2$ into (3) to get

$$|A| + |B| = n \leq |A| + c\,.$$

By the definition of c, it follows that $|B| = c$, and so H is edgeless. Let x be the sole vertex of H'. Since $s = 2$, $d(x) = 2$. Then (1) forces

$$d(y_i) \geq n - 2 \qquad (i = 1, 2)\,, \tag{5}$$

and equality must hold in (5), since y_1, y_2 is an independent set. Therefore, G contains $K_{2,n-2}$ as a spanning subgraph, with $\{y_1, y_2\}$ as one side of the bipartition. The various cases (i), (iii), and (iv) of Theorem 1 follow easily. ∎

Note. H. J. Veldman ([7], Theorem 5) proved that if G is a graph satisfying (1) strictly, then conclusion (i) of Theorem 1 follows when $u = v$.

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