# HOMOMORPHISMS OF 3-CHROMATIC GRAPHS, II

by

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ABSTRACT: This paper presents various attempts to classify which graphs can be homomorphic preimages of the odd cycles. Our principal result is that a graph whose shortest odd cycle has length at least g which does not map to a g-cycle, contains a subgraph which folds to a special homeomorph of a 4-clique. In the case that g = 5 we exhibit the two obstructions and show that deciding whether a graph maps to either of these obstructions is NP-complete.

A homomorphism f from a graph G to a graph H is a mapping from the vertex set of G (denoted by V(G)) to the vertex set of H which preserves edges i.e. if (u,v) is an edge of G, then (f(u),f(v)) is an edge of H. We say G maps to H. If in addition each edge in H is the image of some edge in G, then f is said to be onto and H is called a homomorphic image of G. If G is a graph which has no homomorphism to a proper subgraph of itself, then G is said to be (homomorphism) minimal. Graph colorings provide the most common examples of homomorphisms: an r-coloring of G is just a homomorphism to the r-clique. Those graphs which are critical with respect to coloring are a fortiori homomorphism minimal. While there has been substantial work done on graph homomorphisms by computer scientists working in formal languages and by category theorists (see [5-11,14]), we will be motivated by the idea that homomorphisms are generalizations of colorings.

It has long been realized that the classification of 3-chromatic graphs is a difficult if not hopeless task. Since every such graph must contain an odd cycle it is natural to seek a characterization of the graphs which map to the odd cycles. Maurer, Sudborough, and Welzl have shown that given a fixed odd cycle H, it is NP-complete to decide if a

This paper was originally presented at the 15th Southeastern Conference in Baton Rouge.

Congressus Numerantium, **47** (1985), pp.19-28

graph G has a homomorphism to H [9]. They further conjecture that this remains true as long as H is any minimal graph which contains more than one edge. While this strongly suggests the absence of a good characterization of the preimages of a 5-cycle, there have been three attacks upon this problem prior to our current one which are worthy of note.

Given graphs H and K, the strong product of H and K, denoted by $H \boxtimes K$, is the graph whose vertex set is V(H) x V(K). Adjacency is defined by $(u,v) \sim (x,y)$ if either

i) $u = x$ and $v \sim y$,

ii) $u \sim x$ and $v = y$,

or iii) $u \sim x$ and $v \sim y$.

Vesztergombi used the strong product to characterize preimages of the 5-cycle [12,13]. She showed that a 3-chromatic graph G maps to C, the 5-cycle, if and only if the chromatic number of $G \boxtimes C$ equals 5.

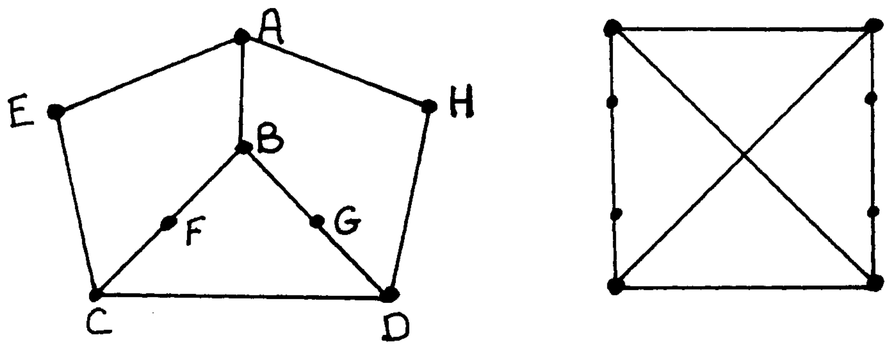
A graph G is said to be H-critical if there does not exist a homomorphism of G to H and if for any edge e there exists a homomorphism from G - e to H. A graph G is said to be TK4 free if G does not contain a subgraph which is homeomorphic to a 4-clique. Catlin has constructively characterized those TK4 free graphs which are 5-cycle-critical [2,4]. He showed that every such graph can be obtained from a triangle by a sequence of three different kinds of replacements, an arc replacement, an edge replacement, and a vertex replacement. In earlier work which foreshadows our first two results, Catlin showed that a 3-cycle- critical graph must contain a TK4 which when embedded in the plane contains only odd faces [3].

Given a graph G, let $\alpha(G,t)$ denote the maximum number of vertices in an induced t-colorable subgraph of G. Albertson and Collins have used these parameters to provide necessary conditions for the existence of a homomorphism [1]. Specifically if there exists a homomorphism from a graph G to a vertex transitive graph H, then for $t = 1,2,\ldots,$

$$\alpha(G,t)/|V(G)| = \alpha(H,t)/|V(H)|.$$

When the target is specified to be the 5-cycle this says that the independence ratio of G must be at least 2/5, the largest induced bipartite subgraph of G must contain at least 4/5 of the vertices of G, and that G must be 3-colorable.

Our first result identifies the obstructions to mapping to the 5-cycle. Clearly one such obstruction is a triangle. In Figure 1 we show two graphs labelled L and P which also cannot be mapped to the 5-cycle. This can be proved directly or by applying the lemma of Albertson and Collins.




L P

Figure 1

THEOREM 1. If G is a triangle free graph which does not map to the 5-cycle, then G contains a subgraph which maps to L or P.

Proof. We prove a slightly stronger version in that we restrict our homomorphisms to be folds. An elementary fold is the identification of two vertices at distance two. A fold is a sequence of elementary folds and is thus a homomorphism. We proceed by induction on the number of vertices. Find a vertex, say x, with neighbors u, v, and w. If no such vertex exists, then G is the union of paths and cycles and trivially maps to the 5-cycle if it does not contain a triangle. Let G' be the graph obtained by folding u onto v. If G' maps to the 5-cycle, then so does G. If G' or a subgraph of it maps to L or P, then so does G or a subgraph. Thus we may assume that G contains a triangle one of whose vertices is the newly created vertex u=v. If x were in the triangle, then G contains a triangle also and there is nothing to prove. Thus we may assume that G' contains a triangle whose vertices are a, b, and u=v.

Returning to G this becomes the 5-cycle x,u,a,b,v. Similar arguments produce the 5-cycles x,w,c,d,u and x,w,e,f,v. Note that the vertices a,b,c,d,e, and f might not all be distinct. If these vertices are all distinct then we can produce L by folding d to a and f to b and we can produce P by folding c to e and then d to f. If these vertices are not all distinct, by symmetry we can focus on just one, say a. If a = c or a = f, then G contains a triangle. If a = e and both d and c are distinct from a and b, then G contains P. If a = e and at least one of d and c are identified with some other vertex in the graph, then G contains a triangle. If a = d and either f = b or c = e, then G contains L. Other identifications produce triangles. If there are no other identified vertices, then folding c to b produces P while folding f to b produces L.

REMARK. If C denotes the 5-cycle, Vesztergombi's theorem implies that neither $C \boxtimes L$ nor $C \boxtimes P$ can be 5-colored. Richard Hughey and Charles Grinstead used their coloring program to 6-color both of these graphs, a result which seemed inaccessible to hand computation.

To generalize the above theorem for larger odd cycles it is necessary to take a closer look at the obstructions L and P. The reader can easily check that each is a subdivision of a 4-clique, what we have referred to above as a TK4. Furthermore, each of the cycles which corresponds to a face in a planar embedding of the 4-clique is a 5-cycle. This inspires the following definition. A TK4 is said to be g-balanced (g an odd integer) if its four faces are each g-cycles. A 3-coloring of the edges of a 4-clique provides a natural pairing of the edges. We call edges which receive the same color matched. In a TK4 two paths will be called matched if the corresponding edges in the 4-clique are matched. By solving four equations in six unknowns we can see that in a g-balanced TK4 the matched paths have the same length. Furthermore the sum of the lengths of the paths equals g. Note that the matched paths in L have lengths 2, 2, and 1 while the matched paths in P have length 3, 1, and 1. These g-balanced TK4's will provide the obstructions to mapping to the g-cycle. That such graphs will not map to the g-cycle follows immediately from the lemma of Albertson and Collins.

THEOREM 2. If $G$ is a graph whose shortest odd cycle has length at least $g$ and $G$ does not map to a $g$-cycle, then $G$ contains a subgraph which folds to a $g$-balanced $TK4$.

Proof. Let $G$ be a minimum counterexample. Then any fold creates a $(g-2)$-cycle. Therefore, every pair of incident edges lies in a $g$-cycle. We may assume that $G$ contains a $g$-cycle $C$ with consecutive vertices $u, x$, and $v$ such that $x$ is adjacent to another vertex $w$ not in $C$.

Let $C'$ denote a $g$-cycle containing $u, x$, and $w$ but not $v$, where $C'$ is chosen to minimize $|E(C') - E(C)|$. We use $E(C)$ to denote the edge set of the cycle $C$. Since $C$ and $C'$ have length $g$ which equals the girth of $G$, either $G - C'$ must be connected or the minimality of $|E(C') - E(C)|$ is violated. We may assume that $u$ is in both $C$ and $C'$. We may also assume that $v, x$, and $w$ are consecutive vertices in a $g$-cycle $C''$. Define $P$ to be $C \cap C'$, and let $t$ denote the end of $P$ opposite $x$. The internally disjoint $(x,t)$ paths of $C \cup C'$ are labelled $P$, $Q$, and $Q'$ where $P$ contains $u$, $Q$ contains $v$, and $Q'$ contains $w$.

By the minimality of $C$ and $C'$, the portion of $C''$ not contained in $C \cup C'$ is a connected path which we call $R''$. Label the vertices of attachment of $R''$ to $C \cup C'$ $y$ and $z$. Since $C''$ contains the edges $(v,x)$ and $(w,x)$, one of these five cases holds:

(i) $C'' = Q \cup Q'$ i. e. $y$ and $z$ do not exist;

(ii) $y$ and $z$ are internal vertices of $Q$;

(iii) $y$ and $z$ are internal vertices of $Q'$;

(iv) either $y$ or $z$ equals $t$;

or (v) $y$ is an internal vertex of $Q$ and $z$ is an internal vertex of $Q'$.

Since $|E(P)| + |E(Q')| = g = |E(P)| + |E(Q)|$,

we have

$$(*) \quad |E(Q)| = |E(Q')|$$

which excludes case (i). Let $R$ and $R''$ denote the two $(y,z)$-segments of the $g$-cycle $C''$ where $R = C'' \cap (C \cup C')$. Since $P \cup Q \cup Q' \cup R''$ has a planar embedding with three faces of odd girth $g$, the fourth face, bounded by $R''$ and by $(Q \cup Q') - R$ must be odd. Therefore,

$$|E(R'')| + |E(Q \cup Q')| - |E(R)| \geq g.$$

Since $|E(R)| + |E(R'')| = g$, we get

$$|E(Q)| + |E(Q')| - 2|E(R)| \geq 0.$$

Therefore, by (*) we obtain

$$(**) \quad |E(Q)| \geq |E(R)| \text{ and } |E(Q')| \geq |E(R)| .$$

The definition of R together with (**) excludes cases (ii), (iii), and (iv). Thus we may assume that the ends y and z are internal vertices of Q and Q' respectively. We embed the TK4, $P \cup Q \cup Q' \cup R''$, in the plane so that x is in the interior and C, C', and C" are three interior faces, all odd cycles of girth g. Since the number of odd faces of an embedded planar graph is even, the outer face must be odd. The outer face can be folded to form a g-cycle obtaining a g-balanced TK4. Theorem 2 follows by contradiction.

COROLLARY 3. The smallest (2k+1)-cycle-critical graph with more than 2k-1 vertices is a (2k+1)-balanced TK4 on 4k vertices.

CONJECTURE. The only (2k+1)-cycle-critical graphs with fewer than 6k vertices are odd cycles and TK4s.

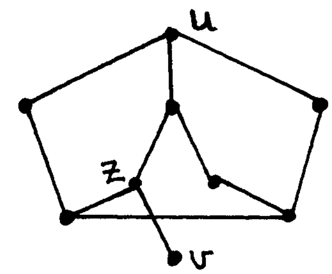
The above conjecture, if true, is best possible. For k at least 2, (2k + 1)-cycle-critical graphs on 6k vertices can be obtained from three disjoint (2k + 1)-cycles by identifying three pairs of appropriately chosen vertices, one pair from each distinct pair of the three cycles. An example is the TK4-free 5-cycle-critical graph with 12 vertices discussed in [2,4].

We now consider the complexity of deciding whether a graph maps to a g-balanced TK4.

THEOREM 4. It is NP-complete to decide if there exists a homomorphism from a graph G onto L.

Proof. Let H[L] denote the graph obtained from H by replacing every edge of H, say (u,v), with a subgraph containing L as shown in Figure 2.


an edge in H




is replaced by this in H[L]

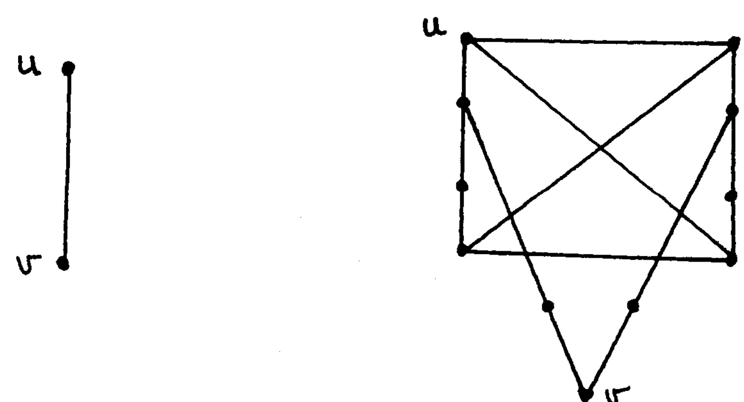
Figure 2

We claim that H is 4-colorable if and only if there exists a homomorphism from H[L] onto L. Thus if there existed a good algorithm for deciding if a graph mapped onto L, then there would exist a good algorithm to test 4-colorability. If there exists a homomorphism from H[L] onto L, then u must be mapped to one of the vertices of degree three. The vertex z must be mapped to a vertex of degree two in L which is not adjacent to the image of u. There exists two choices for the image of z. Then v must be mapped to a vertex of degree three in L which is adjacent to the image of z and hence is not the image of u. If the vertices of degree three in L are labelled as in Figure 1, then we can think of these labels as indicating color classes which correspond with a 4-coloring of H. The other direction is slightly more complicated. Suppose there exists a 4-coloring of H with color classes labelled A, B, C, and D. This coloring when copied to H[L] indicates part of a homomorphism from H[L] to L when L is labelled as in Figure 1. It is necessary to show that this mapping from the vertices of H[L] which are also in H to L extends to a homomorphism of all of H[L] onto L. It suffices to consider only the vertices in H[L] which correspond with one edge in H. There are twelve different cases depending on the coloring of u and v. For example if u is colored A and v is colored B (resp. C), then f(u) = A, f(z) = F, and f(v) = B (resp. C). The remaining vertices in the copy of L which corresponds with the edge (u,v) can clearly be mapped onto L. A slightly harder case occurs when say the color of u is C and the color of v is B. Here we let f(u) = C, f(z) = G, and f(v) = B. That the remaining vertices can be mapped onto L is a consequence of the fact that (AC)(BD) is an automorphism of L. We exhaust the twelve cases in the table below where we indicate f(z) as well as the automorphism of L. Note that in each case f(u) (resp. f(v)) equals the color assigned to u (resp. v) in the coloring of H.

| f(u) = \ f(v) = | A | B | C | D |
|---|---|---|---|---|
| A | *** | f(z) = F<br>Id. | f(z) = F<br>Id. | f(z) = G<br>(CD) |
| B | f(z) = F<br>(AB) | *** | f(z) = E<br>(AB) | f(z) = A<br>(AB)(CD) |
| C | f(z) = H<br>(AC)(BD) | f(z) = G<br>(AC)(BD) | *** | f(z) = A<br>(AC)(BD) |
| D | f(z) = E<br>(AD)(BC) | f(z) = F<br>(AD)(BC) | f(z) = F<br>(AD)(BC) | *** |

THEOREM 5. It is NP-complete to decide if there exists a homomorphism from a graph G onto P.

Proof. Let H[P] denote the graph obtained from H by replacing every edge of H, say (u,v) with a subgraph containing P as shown in Figure 3.




an edge in H

is replaced by this in H[P]

Figure 3

We claim that H is 4-colorable if and only if there exists a homomorphism from H[P] onto P. The proof is similar to the proof of

Theorem 4.

We note that similar techniques suffice to prove the same result for every 7-balanced TK4. It was our hope that the proof would extend to every g-balanced TK4, but as yet we have not been able to accomplish this.

We are grateful to Joan P. Hutchinson for helpful discussions concerning the NP-completeness portions of this paper.

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