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Topological Cliques of Random Graphs

B. BOLLOBÁS AND P. A. CATLIN*

University of Cambridge, Cambridge, England
and Wayne State University, Detroit, Michigan 48202

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Given a graph G, denote by $\text{tcl}(G)$ the largest integer r for which G contains a TK^r, a toplogical complete r-graph. We show that for every $\varepsilon > 0$ almost every graph G of order n satisfies

$$(2-\varepsilon)\, n^{1/2} < \text{tcl}(G) < (2+\varepsilon)\, n^{1/2}.$$

Throughout this paper we follow the notation and terminology of [1]. In particular, $\chi(G)$ is the chromatic number of a graph G, $\Gamma(x)$ is the set of neighbours of a vertex x and TK^r is a topological complete r-graph, that is, a graph homeomorphic to a complete r-graph K^r. We also define the *topological clique number* $\text{tcl}(G)$ *of* G as $\text{tcl}(G) = \max\{r: G \supset TK^r\}$.

A conjecture of Hajós, stating that $\text{tcl}(G) \geqslant \chi(G)$, had been open for over 25 years before Catlin [4] disproved it by exhibiting counterexamples for $\chi(G) \geqslant 7$. Catlin's disproof of this conjecture prompted Erdös and Fajtlowicz [5] to notice that almost every graph is a counterexample to the Hajós conjecture. (For the basic properties of random graphs see [2, Chap.VII].) For it is well known (see [2] or [3] for a sharp result) that for $\varepsilon > 0$ almost every (a.e.) graph G of order n satisfies

$$\left(\frac{\log 2}{2} - \varepsilon\right) \frac{n}{\log n} < \chi(G) < (\log 2 + \varepsilon)\frac{n}{\log n}.$$

On the other hand, it is shown in [5] that for some $c_1 > 0$

$$\text{tcl}(G) < c_1 n^{1/2}$$

for a.e. graph G of order n. Thus

$$\chi(G)/\text{tcl}(G) > c_2 n^{1/2}/\log n$$

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for some positive constant c_2 and a.e. graph G of order n. The aim of this paper is to show that $\text{tcl}(G)$ is about $2n^{1/2}$ for a.e. graph G of order n. Consequently for every $\varepsilon > 0$ and a.e. G,

$$\left(\frac{\log 2}{4} - \varepsilon\right) \frac{n^{1/2}}{\log n} < \chi(G)/\text{tcl}(G) < \left(\frac{\log 2}{2} + \varepsilon\right) \frac{n^{1/2}}{\log n}.$$

THEOREM. *Let $\varepsilon > 0$. Then a.e. graph G of order n satisfies*

$$(2-\varepsilon)\, n^{1/2} < \text{tcl}(G) < (2+\varepsilon)\, n^{1/2}.$$

Furthermore, a.e. G is such that every set W of $m = \lceil (2-\varepsilon)\, n^{1/2} \rceil$ vertices G is the set of branch vertices of a TK^m.

Proof. Stirling's formula has the following easy and well-known consequence. The probability that after n tosses of an unbiased coin the difference between heads and tails is at least εn is not more than $(2/\varepsilon)\, e^{-\varepsilon^2 n/2}$. Similarly, if heads occur with probability p, $0 < p < 1$, then the probability of having less than $(p-\varepsilon)n$ heads or more than $(p+\varepsilon)n$ heads is at most e^{-cn}, where $c > 0$ depends only on p and ε. This allows us to deduce the following simple properties of almost all graphs.

(i) *If $\varepsilon > 0$ is fixed and $m/\log n \to \infty$ then a.e. graph is such that every subgraph H spanned by m vertices satisfies*

$$(\tfrac{1}{4} - \varepsilon)\, m^2 \leqslant e(H) \leqslant (\tfrac{1}{4} + \varepsilon)\, m^2.$$

Indeed, the probability that a given H fails to satisfy the inequalities is at most $e^{-\varepsilon^2 m^2}$ provided n is large enough. There are $\binom{n}{m}$ choices for H and as $n \to \infty$,

$$\binom{n}{m} e^{-\varepsilon^2 m^2} \leqslant n^m e^{-\varepsilon^2 m^2} \to 0.$$

(ii) *Given $k \in N$ and $\delta > 0$, a.e. graph G is such that whenever $x_1, x_2, \ldots, x_{2k}$ are distinct vertices,*

$$\left| \bigcup_{i=1}^{k} (\Gamma(x_{2i-1}) \cap \Gamma(x_{2i})) \right| \geqslant (1 - (\tfrac{3}{4})^k - \delta)n. \qquad (*)$$

Indeed, let $x_1, x_2, \ldots, x_{2k}$ be fixed and choose a vertex $x \in V(G) - \{x_1, x_2, \ldots, x_{2k}\}$. The probability that for a given i the vertex x belongs to $\Gamma(x_{2i-1}) \cap \Gamma(x_{2i})$ is $\frac{1}{4}$, so the probability that x belongs to $\bigcup_{i=1}^{k} (\Gamma(x_{2i-1}) \cap \Gamma(x_{2i}))$ is $1 - (\frac{3}{4})^k$. Therefore the probability of $(*)$ failing is at most e^{-cn} for some constant $c > 0$. Since there are fewer than n^{2k} choices for $x_1, x_2, \ldots, x_{2k}$, and

$$n^{2k} e^{-cn} \to 0 \qquad \text{as} \quad n \to \infty,$$

the assertion follows.

The theorem is now easily proved

(a) Suppose G contains a TK^m whose set of branch vertices is W. If $x, y \in W$ are non-adjacent then the topological $x-y$ edge (which is n $x-y$ path) contains at least one vertex not in W. As the topological edges are disjoint, in $G[W]$ at most $n-m$ edges are missing. We know from (i) that a.e. graph is such that from every subgraph spanned by $\lfloor (2+\varepsilon) n^{1/2} \rfloor$ vertices more than n edges are missing. Hence almost no graph contains a TK^m with $m = \lfloor (2+\varepsilon) n^{1/2} \rfloor$.

(b) Let $0 < \varepsilon < \frac{1}{4}$, $m = \lfloor (2-\varepsilon) n^{1/2} \rfloor$ and choose $k_0 \in \mathbb{N}$ so that $(\frac{3}{4})^{k_0} \in \varepsilon/2$. Put $\delta = \varepsilon/2 - (\frac{3}{4})^{k_0}$. Almost every graph is such that from every subgraph spanned by m vertices less than $(1-(\varepsilon/2))n$ edges are missing and $(*)$ is satisfied for $k \leqslant k_0$.

Let G be a graph with the properties above and let W be any set of m vertices of G. Let $F \subset W^{(2)}$ be the set of pairs of non-adjacent vertices of W. (Thus F is the set of non-edges of $G[W]$.) By our choice of G we have $|F| < (1-\varepsilon/2)n$. Define a bipartite graph with vertex classes F and $Z = V(G) - W$ by joining $xy \in F$ to $Z \in Z$ if z is a common neighbour of x and y. Note that every edge of B corresponds to a path of length 2 joining two vertices of W. In order to show that W is the set of branch vertices of a TK^m, it suffices to show that B has a matching from F into Z, for then a TK^m can be obtained by adding appropriate paths of length 2 to $G[W]$.

To complete the proof we have to check only that the condition of Hall's theorem [6] (see also [1, pp. 9 and 52]) is satisfied. If a set $F' \subset F$ contains a set F'' of k_0 independent non-edges, then by $(*)$ the graph B satisfies $|\Gamma(F')| \geqslant |\Gamma(F'')| \geqslant (1-(\frac{3}{4})^{k_0} - \delta)n = (1-\varepsilon/2)n \geqslant |F| \geqslant |F'|$. On the other hand, if F' does not contain k_0 independent non-edges then trivially

$$|F'| \leqslant 2k_0 m$$

(for sharper estimates see [1, p. 58]). To estimate $\Gamma(F')$ all we need to note is that if $x_1 x_2 \in F'$ then by $(*)$ applied with $k=1$, for every sufficiently large n we have

$$\Gamma(F') \geqslant |\Gamma(x_1) \cap \Gamma(x_2)| \geqslant (\tfrac{1}{4} - \delta)n \geqslant 2k_0 m \geqslant |F'|. \quad \blacksquare$$

The theorem can easily be carried over to the case when the edges of G are chosen independently and with a fixed probability p, that is, if we consider the probability space $\mathscr{G}(n, P(\text{edge}) = p)$ of [2, p. 123]. For every $\varepsilon > 0$ a.e. $G \in \mathscr{G}(n, P(\text{edge}) = p)$ satisfies

$$\left\{\left(\frac{2}{1-p}\right)^{1/2} - \varepsilon\right\} n^{1/2} < \mathrm{tcl}(G) < \left\{\left(\frac{2}{1-p}\right)^{1/2} + \varepsilon\right\} n^{1/2}.$$

In conclusion we note another related result. Denote by $TK^{m:s}$ a topological K^m obtained from a K^m by subdividing every edge into exactly s edges. Define $k = k(s, n)$ to be the maximal integer satisfying

$$k + \binom{k}{2}(s-1) \leqslant n.$$

Thus k is the maximal integer for which K^n contains a $TK^{k:s}$. A slightly more complicated version of the last part of the proof (the application of Hall's theorem) gives the following result.

Let $s \geqslant 2$ be fixed. Then for a.e. $G \in \mathscr{G}(n, P(\text{edge}) = p)$ we have

$$\max\{m : G \supset TK^{m:s}\} = k(s, n).$$

Furthermore, a.e. G is such that every set of k vertices is the set of branch vertices of some $TK^{k:s}$.

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