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Hadwiger's Conjecture is True for Almost Every Graph

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The contraction clique number ccl(G) of a graph G is the maximal r for which G has a subcontraction to the complete graph K^r. We prove that for $d>2$, almost every graph of order n satisfies $n((\log_2 n)^{\frac{1}{2}}+4)^{-1} \leq \text{ccl}(G) \leq n(\log_2 n - d \log_2 \log_2 n)^{-\frac{1}{2}}$. This inequality implies the statement in the title.

1. INTRODUCTION

One of the deepest unsolved problems in graph theory is the following conjecture due to Hadwiger [7]: $\chi(G)=s$ implies $G>K^s$. In other words, every s-chromatic graph G has a subcontraction to K^s, the complete graph of order s. In the case $s=5$, this is equivalent to the four-colour theorem. (For an account of the various results related to Hadwiger's conjecture the reader is referred to [1, Chapter VII]; the terminology and notation not defined here can also be found in [1].)

The statement in the title would sound rather hollow but for certain recent developments. Hajós conjectured that every s-chromatic graph contains a TK^s, a topological complete subgraph of order s, that is a subdivision of K^s. This is clearly stronger than Hadwiger's conjecture, for a TK^s itself has a contraction to K^s, but a graph subcontractible to K^s need not contain a TK^s. The Hajós conjecture was disproved recently by Catlin [5], who exhibited counter-examples for $\chi(G)\geq 7$. Shortly after Catlin's result Erdös and Fajtlowicz [6] showed that almost every graph is a counter-example to the Hajós conjecture. More precisely, define the *topological clique number* of a graph G as

$$\text{tcl}(G)=\max\{r\colon G\supset TK^r\}.$$

Erdös and Fajtlowicz showed that for almost every graph G of order n,

$$\text{tcl}(G)\leq cn^{\frac{1}{2}} \tag{1}$$

for some absolute constant c. Since for every $\varepsilon>0$ almost every graph satisfies

$$\chi(G)\geq(\tfrac{1}{2}-\varepsilon)n/\log_2 n,$$

we have that

$$\text{tcl}(G)<\chi(G)$$

for almost every graph (for sharp results on $\chi(G)$ see [4]).

Inequality (1) was extended by Bollobás and Catlin [3], who proved that for every $\varepsilon>0$ almost every graph satisfies

$$(2-\varepsilon)n^{\frac{1}{2}}\leq \text{tcl}(G)\leq(2+\varepsilon)n^{\frac{1}{2}} \tag{2}$$

and so

$$(\tfrac{1}{4}-\varepsilon)n^{\frac{1}{2}}/\log_2 n\leq\chi(G)/\text{tcl}(G).$$

In view of this it is imperative to attack Hadwiger's conjecture by random graphs, that is to examine whether or not Hadwiger's conjecture holds for almost every graph. This is

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exactly the task we shall accomplish in this note. More precisely, we shall prove an analogue of (2) for the *contraction clique number* ccl(G) of a graph G, defined as

$$\mathrm{ccl}(G) = \max\{r \colon G > K^r\}.$$

2. Random Graphs

Let $0<p<1$ be fixed, and let V be a set of n distinguishable vertices. Denote by $\mathcal{G}(n, P(\mathrm{edge}) = p)$ the discrete probability space consisting of all graphs with vertex set V, in which the probability of a graph of size m is

$$p^m(1-p)^{\binom{n}{2}-m}.$$

In other words, the edges of a graph $G \in \mathcal{G}(n, P(\mathrm{edge}) = p)$ are chosen independently and with probability p. (See [2, Chapter VII] for results concerning this model.)

Given a property $\mathcal{P}$ of graphs we define the *probability of* $\mathcal{P}$ as

$$P(\mathcal{P}) = P(\{G \in \mathcal{G}(n, P(\mathrm{edge}) = p) \colon \mathcal{P} \text{ holds for } G\}).$$

If $P(\mathcal{P}) \to 1$ as $n \to \infty$ then the property $\mathcal{P}$ is said to hold for *almost every* graph.

In order to make the calculations below a little more pleasant, we shall take $p=\frac{1}{2}$. The case $p=\frac{1}{2}$ is in some sense the most natural, since this is the model one considers implicitly when one counts the proportion of all graphs having a given property. Indeed, in the model $\mathcal{G} = \mathcal{G}(n, P(\mathrm{edge}) = \frac{1}{2})$ every graph has the same probability, so the probability of a set $\mathcal{H} \subset \mathcal{G}$ is exactly $|\mathcal{H}|/|\mathcal{G}|$. Thus a property $\mathcal{P}$ holds for almost every graph in $\mathcal{G}(n, P(\mathrm{edge}) = \frac{1}{2})$ iff the number of graphs having $\mathcal{P}$ is asymptotically equal to the number of all graphs (with vertex set V).

3. The Contraction Clique Number

Given a graph G and non-empty disjoint subsets $V_1, V_2, \ldots, V_s$ of $V = V(G)$, denote by $G/\{V_1, \ldots, V_s\}$ the graph with vertex set $\{V_1, V_2, \ldots, V_s\}$ in which V_i is joined to V_j iff G contains a $V_i - V_j$ edge. Put

$$\mathrm{ccl}'(G) = \max\{r \colon G/\{V_1, \ldots, V_r\} \cong K^r \text{ for some } V_1, \ldots, V_r\}.$$

Since the contraction clique number is defined similarly, except with the added restriction on the V_i that each $G[V_i]$ is connected,

$$\mathrm{ccl}(G) \leq \mathrm{ccl}'(G).$$

We shall give a lower bound for ccl(G) and an upper bound for ccl$'(G)$ holding for almost every graph. As customary, $\log_b x$ denotes the logarithm to base b.

Theorem. *Let $d>2$. Then almost every graph $G \in \mathcal{G}(n, P(\mathrm{edge}) = \frac{1}{2})$ satisfies*

$$n((\log_2 n)^{\frac{1}{2}}+4)^{-1} \leq \mathrm{ccl}(G) \leq \mathrm{ccl}'(G)$$
$$\leq n(\log_2 n - d\log_2\log_2 n)^{-\frac{1}{2}} \leq n((\log_2 n)^{\frac{1}{2}}-1)^{-1}.$$

Proof. (a) We start with a proof of the upper bound on ccl$'(G)$. Put $s = \lfloor n(\log_2 n - d\log_2\log_2 n)^{-\frac{1}{2}}\rfloor$. A partition $\{V_1, V_2, \ldots, V_s\}$ of the vertex set V is said to be *permissible for a graph G* if G contains a $V_i - V_j$ edge for every pair (i,j), $1 \leq i < j \leq s$. Then ccl$'(G) \geq s$ iff the graph G has a permissible partition. We have to prove that the probability that a graph has a permissible partition tends to 0 as $n \to \infty$.

To start with, note rather crudely that there are at most

$$\frac{n!}{s!}\binom{n}{s-1} < n^n \tag{3}$$

partitions of V into s non-empty sets. The number on the left-hand side of (3) is the number of partitions of V into s non-empty *ordered* sets.

Consider now a fixed partition $\mathscr{P} = \{V_1, V_2, \ldots, V_s\}$ into non-empty sets. What is the probability that this partition $\mathscr{P}$ is permissible? Let $n_1, n_2, \ldots, n_s$ be the number of vertices in the classes. Then the probability that a graph contains no $V_i - V_j$ edge is $2^{-n_i n_j}$. Hence

$$P(\mathscr{P} \text{ is permissible}) = \Pi(1-2^{-n_i n_j}) \leqslant e^{-\Sigma 2^{-n_i n_j}}, \tag{4}$$

where both the product and the sum are taken over all pairs (i, j) with $1 \leqslant i < j \leqslant s$. We have the following string of elementary inequalities.

$$\Sigma 2^{-n_i n_j}\binom{s}{2}^{-1} \geqslant (\Pi 2^{-n_i n_j})^{\binom{s}{2}^{-1}} = 2^{-(\Sigma n_i n_j)\binom{s}{2}^{-1}} \geqslant 2^{-n^2/s^2}. \tag{5}$$

The reader may note that $\Sigma n_i n_j$ is exactly the number of edges in the complete s-partite graph with vertex classes $V_1, V_2, \ldots, V_s$. The Turán graph $T_s(n)$ is the unique s-partite graph with maximal number of edges, and

$$e(T_s(n)) = \left(\frac{s-1}{2s} + o(1)\right) n^2 \qquad \text{(see [2, p. 71])}.$$

From (4) and (5) we have

$$P(\mathscr{P} \text{ is permissible}) \leqslant e^{-\binom{s}{2}2^{-n^2/s^2}}, \tag{6}$$

and (3) and (6) imply

$$P(G \text{ has a permissible partition} = P(\mathrm{ccl}'(G) \geqslant s) \leqslant n^n e^{-\binom{s}{2}2^{-n^2/s^2}}$$
$$= P_s. \tag{7}$$

Clearly

$$\log P_s = n \log n - \binom{s}{2} 2^{-n^2/s^2} \leqslant n\left\{\log n - \frac{1}{3\log_2 n} 2^{d \log_2 \log_2 n}\right\} \leqslant -\tfrac{1}{4}n(\log_2 n)^{d-2} \to -\infty.$$

Hence $P_s \to 0$, proving the required upper bound on $\mathrm{ccl}'(G)$.

(b) We turn to the proof of the lower bound on $\mathrm{ccl}(G)$. Put $k = \lceil (\log n)^{\frac{1}{2}} + \frac{1}{2} \rceil$, $s = \lceil n/(k^5/2) \rceil$ and $t = \lfloor n/(k+2) \rfloor$. We shall prove in two steps that $G > K^s$ for almost every graph G.

Step 1. Fix a set T of t vertices and put $W = V - T$. Then *almost every graph G contains t vertex disjoint stars of order $k+1$ whose centres are the t vertices in T.*

Indeed, by a slight extension of Hall's theorem (see [2, p. 56]) if G does not contain such stars then there is a set $A \subset T$ for which the vertices in A have less than $k|A|$ neighbours in W. Given a set A with $a = |A|$ elements, the probability that a vertex in W is joined to no vertex in A is 2^{-a}. Hence the probability that the vertices in A have less than ka neighbours in W is at most

$$\sum_{u<ka} \binom{n-t}{u} 2^{-a(n-t-u)} < n^{ka} 2^{-a(n-t-ka)}$$
$$\leqslant n^{ka} 2^{-at} < 2^{-at/2}.$$

Consequently the probability that G fails to contain the desired t stars is at most

$$\sum_{a\le t}\binom{t}{a}2^{-at/2}\le\sum_{a\le t}(t2^{-t/2})^a\le 2t2^{-t/2},$$

and this tends to 0.

Step 2. Let $V_1, V_2, \ldots, V_t$ be the vertex sets of the stars constructed in Step 1 in almost every graph. Then for *almost every graph G there are $V_{n_1}, V_{n_2}, \ldots, V_{n_s}$ such that $G/\{V_{n_1}, V_{n_2}, \ldots, V_{n_s}\}\cong K^s$.* The assertions in these two steps clearly imply the first inequality of our theorem.

Note that the sets $V_1, V_2, \ldots, V_t$ depend only on the $T-W$ edges of the graph. Thus the edges joining the vertices of W are chosen independently with probability $\frac{1}{2}$. Put $W_i = V_i - T$. We say that (W_i, W_j), $i\neq j$, is *good* if there is a $W_i - W_j$ edge. Since $W_i\subset W$ and $|W_i| = k$, clearly

$$P(\text{the pair } (W_i, W_j) \text{ is bad}) = 2^{-k^2}$$

and so the expected number of bad pairs is

$$\binom{t}{2}2^{-k^2}<\frac{n^2}{\log_2 n}2^{-\log_2 n-(\log_2 n)^{\frac{1}{2}}}=\frac{n}{\log_2 n}2^{-(\log_2 n)^{\frac{1}{2}}}.$$

At this stage we have several options. We may appeal either to the classical De Moivre–Laplace theorem (see [2; p. 134]) or to the even simpler Chebyshev inequality (see [2, p. 134]) or to the trivial inequality $P(|X|\ge|c|)\le E(|X|)/|c|$ to deduce that almost every graph has few bad pairs. For example, the last inequality implies that the probability that a graph has more than

$$\frac{n}{\log_2 n}2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}}$$

bad pairs is at most $2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}}$. In particular, since

$$t-\frac{n}{\log_2 n}2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}}>s,$$

for almost every graph we can find sets $W_{n_1}, W_{n_2}, \ldots, W_{n_s}$ such that every pair (W_{n_i}, W_{n_j}) is good. Then we have $G/\{V_{n_1}, \ldots, V_{n_s}\}\cong K^s$ and since each $G[V_i]$ is connected, $\mathrm{ccl}(G)\ge s$, as claimed.

The proof of our theorem is complete.

With a little more effort the lower bound can be improved to $n((\log_2 n)^{\frac{1}{2}}+1)^{-1}$. Furthermore, the calculations can easily be carried over to the general case. If $0<p<1$ is fixed then almost every graph in $\mathcal{G}(n, P(\text{edge})=p)$ satisfies the inequality in the Theorem, with $\log_2 n$ replaced by $\log_b n$, where $b=1/q$.

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