Graph Decompositions Satisfying Extremal Degree Constraints*

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ABSTRACT

We show that the vertex set of any graph G with $p \geq 2$ vertices can be partitioned into non-empty sets V_1, V_2, such that the maximum degree of the induced subgraph $\langle V_i \rangle$ does not exceed

$$\frac{p_i - 1}{p - 1} \Delta(G),$$

where $p_i = |V_i|$, for $i = 1, 2$. Furthermore, the structure of the induced subgraphs is investigated in the extreme case.

1. INTRODUCTION

We consider the problem of partitioning the vertex set of a graph G, satisfying a certain constraint upon the maximum degree $\Delta(G)$ of its vertices, into sets V_1, V_2 having p_1 and p_2 vertices, respectively, such that the induced subgraphs G_1 and G_2 satisfy a similar constraint upon the maximum degree of their vertices.

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We shall follow the notation of Harary [1]. Also, we denote by $E(V_1, V_2)$ the set of edges of G with one end in V_1 and the other end in V_2.

The following theorem, essentially due to Lovász [2], is the sort in which we are interested.

Theorem 1.1. Let G be a graph with $\Delta(G)=h$ for some $h\geq 1$. Let h_1, h_2 be non-negative integers that satisfy

$$h=h_1+h_2+1.$$

Any partition $V_1\cup V_2$ of $V(G)$ that maximizes

$$f_1(V_1, V_2)=|E(V_1, V_2)|+h_1\,|V_1|+h_2\,|V_2|$$

is nontrivial and satisfies

$$\Delta(G_i)\leq h_i, \qquad i=1,2.$$

Lovász maximized an expression different than $f_1(V_1, V_2)$. However, the proof of Theorem 1.1 is virtually the same.

As Lovász remarked, Theorem 1.1 may be inductively applied to the induced subgraphs G_i, and a decomposition into numerous induced subgraphs is obtained. A similar remark applies to Theorem 2.1 below.

We shall first prove a theorem analogous to Theorem 1.1, but in which the bound on $\Delta(G_i)$ depends upon p_i.

2. MAIN RESULTS

Define, for $c\in(0, 1]$, $f_2(V_1, V_2)=|E(V_1, V_2)|+\frac{1}{2}\,cp_1^2+\frac{1}{2}\,cp_2^2$.

Theorem 2.1. Let G be a graph with

$$\Delta(G)=c(p-1)$$

for $c\in(0, 1]$ and $p\geq 2$. For any partition $V_1\cup V_2$ of $V(G)$ with

$$f_2 \quad \text{maximized,} \tag{2.1}$$

and

$$\tfrac{1}{2}\,c(p_1^2+p_2^2) \quad \text{minimized, subject to (2.1)} \tag{2.2}$$

it follows that

$$V_1 \cup V_2 \quad \text{is a nontrivial partition;} \tag{2.3}$$

and for $i = 1, 2$,

$$\Delta(G_i) \leq c(p_i - 1). \tag{2.4}$$

Proof. Define the linear function

$$c(t) = c - t, \tag{2.5}$$

where $t \geq 0$. Thus,

$$\Delta(G) = c(p-1) = c(t)(p-1) + t(p-1). \tag{2.6}$$

For any partition $V_1 \cup V_2$ of $V(G)$ and any $t \geq 0$, define

$$F_t(V_1, V_2) = |E(V_1, V_2)| + \tfrac{1}{2}\, c(t)(p_1^2 + p_2^2). \tag{2.7}$$

Thus, for V_1 and V_2 fixed, F_t is a linear function of t with F-intercept $f_2(V_1, V_2)$ and with slope $-\frac{1}{2}(p_1^2 + p_2^2)$. Moreover, F_0 is equal to f_2.

Therefore, if $V_1 \cup V_2$ satisfies (2.1), then for any other partition $U_1 \cup U_2$ of $V(G)$,

$$F_0(V_1, V_2) \geq F_0(U_1, U_2).$$

Also, (2.2) assures that if $U_1 \cup U_2$ is another partition that maximizes $f_2(V_1, V_2)$, then

$$F_t(V_1, V_2) \geq F_t(U_1, U_2).$$

Thus, the only way that we could have

$$F_t(V_1, V_2) < F_t(U_1, U_2)$$

if (2.1) and (2.2) hold is if

$$F_0(V_1, V_2) > F_0(U_1, U_2)$$

and if the slope of $F_t(V_1, V_2)$ is strictly less than that of $F_t(U_1, U_2)$, and t is sufficiently large. Thus, for $t \geq 0$ sufficiently close to 0, if (2.1) and (2.2) hold, then $V_1 \cup V_2$ also maximizes F_t. We shall consider t to be small enough so that $V_1 \cup V_2$ also maximizes F_t.

Reversing the indices if necessary, we may suppose without loss of generality that V_1 is nonempty. Let $v \in V_1$. We have

$$\begin{aligned} 0 &\le F_t(V_i, V_2) - F_t(V_1 - v, V_2 + v) \\ &= |E(V_1, V_2)| + \tfrac{1}{2}c(t)(p_1^2 + p_2^2) \\ &\quad - |E(V_1 - v, V_2 + v| - \tfrac{1}{2}c(t)((p_1 - 1)^2 + (p_2 + 1)^2) \\ &= |E(v, V_2)| - |E(v, V_1)| + c(t)p_1 - c(t)p_2 - c(t). \end{aligned} \tag{2.8}$$

We add $2\deg_{G_1}(v) = 2\,|E(v, V_1)|$ to each side and get

$$\begin{aligned} 2\deg_{G_1}(v) &\le \deg_G(v) + c(t)p_1 - c(t)p_2 - c(t) \\ &\le (c(t) + t)(p_1 + p_2 - 1) + c(t)p_1 - c(t)p_2 - c(t) \\ &= 2c(t)(p_1 - 1) + t(p - 1). \end{aligned}$$

We divide by 2 and substitute for $c(t)$ to get

$$\begin{aligned} \deg_{G_1}(v) &\le c(t)(p_1 - 1) + \tfrac{1}{2}t(p - 1) \\ &= c(p_1 - 1) - t(p_1 - 1) + \tfrac{1}{2}t(p - 1) \\ &= c(p_1 - 1) + \tfrac{1}{2}t(p - 2p_1 + 1). \end{aligned} \tag{2.9}$$

If $G_1 = G$, then $p_1 = p$, whence by (2.9), if v is a vertex of maximum degree in G, then

$$\begin{aligned} \deg_G(v) &= \deg_{G_1}(v) \\ &\le c(p - 1) + \tfrac{1}{2}t(1 - p) \\ &< c(p - 1) \\ &= \deg_G(v), \end{aligned}$$

a contradiction. Hence, (2.3) holds, and (2.9) applies to either set V_1 or V_2. Since (2.9) holds for $t = 0$, (2.4) follows. ∎

Let $V_1 \cup V_2$ be a nontrivial partition that maximizes $f_j(V_1, V_2)$, with $j = 1$ in Theorem 1.1 or with $j = 2$ in Theorem 2.1. If Theorem 2.1 applies, assume also that (2.2) holds. If $v_1 \in V_1$ and $v_2 \in V_2$ have the property that

$$|E(V_1, V_2)| = |E(V_1 + v_2 - v_1, V_2 + v_1 - v_2)|, \tag{2.10}$$

then $(V_1+v_2-v_1)\cup(V_2+v_1-v_2)$ is also a partition of $V(G)$ such that the above conditions hold. Any pair v_1, v_2 of vertices satisfying condition (2.10) are called *interchangeable.* If $v_1\in V_1$ and $v_2\in V_2$ are interchangeable vertices, then $G[V_1+v_2-v_1]$ and $G[V_2+v_1-v_2]$ satisfy the same conclusions in Theorems 1.1 and 2.1 that apply to $G[V_1]$ and $G[V_2]$.

Theorem 2.2. If in Theorem 1.1 or 2.1, $v_1\in V_1$ and $v_2\in V_2$ are two adjacent vertices such that

$$\deg_{G_1}(v_1)+\deg_{G_2}(v_2)=\Delta(G)-1, \tag{2.11}$$

then v_1 and v_2 are interchangeable, and we have for $i=1, 2$,

$$\deg_{G_i}(v_i)=\begin{cases} h_i & \text{in Theorem 1.1} \\ [c(p_i-1)] & \text{in Theorem 2.1} \end{cases}$$

and

$$\deg_G(v_i)=\Delta(G).$$

If v_3 is another vertex that is interchangeable with v_1, then v_2 and v_3 are adjacent in G.

Proof. Let $v_1\in V_1$ and $v_2\in V_2$ be adjacent vertices satisfying (2.11), where $V_1\cup V_2$ is a partition of $V(G)$ that maximizes $f_1(V_1, V_2)$ in Theorem 1.1 or maximizes $f_2(V_1, V_2)$ and satisfies (2.2) in Theorem 2.1. We have

$$\begin{aligned}
&|E(V_1+v_2-v_1, V_2+v_1-v_2)| \\
&\quad=|E(V_1, V_2)|+\deg_{G_1}(v_1)+\deg_{G_2}(v_2)-|E(v_1, V_2-v_2)|-|E(v_2, V_1-v_1)| \\
&\quad=|E(V_1, V_2)|+2\deg_{G_1}(v_1)+2\deg_{G_2}(v_2)-|E(v_1, V(G)-v_2)| \\
&\qquad\qquad -|E(v_2, V(G)-v_1)| \\
&\quad=|E(V_1, V_2)|+2(\Delta(G)-1)-(\deg_G(v_1)-1)-(\deg_G(v_2)-1) \\
&\qquad\qquad [\text{by } (2.11)] \\
&\quad\geq|E(V_1, V_2)|. \qquad (2.12)
\end{aligned}$$

By the maximality of $f_i(V_1, V_2)$ in Theorems 1.1 and 2.1, $|E(V_1, V_2)|$ cannot be less than $|E(V_1+v_2-v_1, V_2+v_1-v_2)|$. Hence, (2.12) holds with equality. Thus, v_1 and v_2 are interchangeable. Also, since (2.12) holds with equality,

$$\Delta(G)-1=\deg_G(v_i)-1, \qquad i=1,2,$$

whence,

$$\deg_G (v_i) = \Delta(G).$$

Observe that if (2.11) holds, then $\deg_{G_1} (v_1)$ and $\deg_{G_2} (v_2)$ attain the upper bound specified by Theorem 1.1 or 2.1, whichever is applicable. For instance, from (2.11) and from (2.4) of Theorem 2.1,

$$\begin{aligned}
\Delta(G)-1 &= \deg_{G_1} (v_1)+\deg_{G_2} (v_2) \\
&\leq \Delta(G_1)+\Delta(G_2) \\
&\leq c(p_1-1)+c(p_2-1) \\
&= c(p-1)-c \\
&= \Delta(G)-c \\
&< (G).
\end{aligned}$$

Thus, since $\Delta(G)$ is an integer,

$$\deg_{G_i} (v_i) = \Delta(G_i) = [c(p_i-1)],$$

for $i=1$ and 2. In Theorem 1.1, we can more easily obtain

$$\deg_{G_i} (v_i) = h_i \qquad i=1,2.$$

If, contrary to the conclusion of Theorem 2.1, v_2 is not adjacent to v_3, then in $G[V_2+v_1-v_3]$, v_2 is adjacent to v_1 and to h_2 or $[c(p_2-1)]$, respectively, other vertices in $G[V_2+v_1-v_3]$, depending upon whether we consider Theorem 1.1 or Theorem 2.1, respectively. However, we have

$$\Delta(G[V_2+v_1-v_3]) \leq \begin{cases} h_2 & \text{in Theorem 1.1} \\ [c(p_2-1)] & \text{in Theorem 2.1} \end{cases}$$

since v_1 and v_2 are interchangeable, and so we have a contradiction. Thus, v_2 must be adjacent to v_3. ∎

References

[1] F. Harary, *Graph Theory*. Addison-Wesley, Reading, Mass. (1969).

[2] L. Lovász, On decomposition of graphs. *Studia Sci. Math. Hungarica* 1 (1966) 237–238.