Discrete Mathematics 22 (1978) 81–83.

NOTE

A BOUND ON THE CHROMATIC NUMBER OF A GRAPH*

Paul A. CATLIN
Wayne State University, Detroit, MI 48202, U.S.A.

Received 25 May 1976
Revised 13 December 1976

We give an upper bound on the chromatic number of a graph in terms of its maximum degree and the size of the largest complete subgraph. Our result extends a theorem due to Brooks.

1. Main result

Let G be a finite graph with no loops or multiple edges. Let $\chi(G)$ denote the chromatic number of G and $\Delta(G)$ denote the maximum degree of the vertices of G. For a subset X of the set $V(G)$ of vertices of G, let $G[X]$ denote the subgraph of G induced by X. If every pair of vertices of X is adjacent then $G[X]$ is called a complete subgraph of G.

It is well-known that we always have

$$\chi(G) \leq \Delta(G)+1. \tag{1}$$

If G contains a complete subgraph on $\Delta(G)+1$ vertices then, obviously, the equality in (1) must hold. The following basic theorem is due to Brooks [1].

Theorem 1.1. *Let G be a graph with $\Delta(G) \geq 3$. If G does not contain a complete subgraph on $\Delta(G)+1$ vertices, then*

$$\chi(G) \leq \Delta(G). \tag{2}$$

Our main result in this paper is the following:

Theorem 1.2. *Let G be any graph. If G does not contain a complete subgraph on r vertices where $4 \leq r \leq \Delta(G)+1$, then*

$$\chi(G) \leq \Delta(G)+1-\left[\frac{\Delta(G)+1}{r}\right]. \tag{3}$$

* This research is part of the author's Ph.D. Thesis done at the Ohio State University under Professor Neil Robertson.

We observe that if $r = \Delta(G)+1$ then Theorem 1.2 reduces to Theorem 1.1.

Proof. Let $n = [(\Delta(G)+1)/r]$. Clearly, $n \geq 1$. We define integers $h_1, h_2, \ldots, h_n$ as follows: $h_i = r-1$ for $1 \leq i \leq n-1$ and $h_n = \Delta(G) - r(n-1)$. The integers h_i are nonnegative and $\sum_{i=1}^{n} h_i = \Delta(G) - n + 1$. Now, by a theorem due to Lovász [2], there exists a partition of $V(G)$ into n sets $X_1, X_2, \ldots, X_n$ such that

$$\Delta(G[X_i]) \leq h_i = r-1 \quad \text{for } i = 1, 2, \ldots, n-1,$$

$$\Delta(G[X_n]) \leq h_n = \Delta(G) - r(n-1).$$

Since G contains no complete subgraph on r vertices, neither do the subgraphs $G[X_i]$ for $1 \leq i \leq n$. Hence, we can easily see, using Theorem 1.1, that

$$\chi(G[X_i]) \leq r-1 \quad \text{for } i = 1, 2, \ldots, n-1,$$

$$\chi(G[X_n]) \leq \Delta(G) - r(n-1).$$

The last inequality follows because, by definition of n, $\Delta(G) - r(n-1) \geq r-1$, and so Theorem 1.1 may be applied to $G[X_n]$ as well.

Finally,

$$\chi(G) \leq \sum_{i=1}^{n} \chi(G[X_i]) \leq (n-1)(r-1) + \Delta(G) - r(n-1) = \Delta(G) + 1 - n$$

and the theorem is proved.

Theorem 1.2 may be restated as follows:

Theorem 1.3. *Let G be any graph. If G does not contain a complete subgraph on r vertices where $4 \leq r \leq \Delta(G)+1$, then*

$$\Delta(G) \geq \frac{r}{r-1}\chi(G) - 2. \tag{4}$$

2. Remarks

Let $\theta(G)$ denote the maximum number of vertices in a complete subgraph of G. $\theta(G)$ is called the clique number of G.

If $\chi(G) = \Delta(G)$, i.e., if the equality in Theorem 1.1. holds, then by (4), Theorem 1.3, we easily deduce

$$\Delta(G) \geq \theta(G) \geq [\tfrac{1}{2}(\Delta(G)+1)]. \tag{5}$$

We do not know if the lower bound for $\theta(G)$ in (5) can be attained in general.

If $\theta(G) \leq 3$, then as a special case of Theorem 1.2, taking $r = 4$, we obtain

In particular, if $\theta(G) \leqslant 2$, i.e., if G contains no triangles, then obviously (6) still holds. However, in this case it would be natural to expect that the bound (6) can be improved. More generally, the bound (3) or (4) may itself be improvable. In fact, we know of no example of a graph G for which $\chi(G) < \Delta(G)$ and the equality in (3), Theorem 1.2, for instance, holds.

It has recently come to our attention that O.V. Borodin and A.V. Kostochka have independently obtained Theorem 1.2. Their result appears in a preprint titled "On an upper bound of the graph's chromatic number depending on graph's degree and density".

References

[1] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.
[2] L. Lovász, On decomposition of graphs, Studia Sci. Math. Hungar. 1 (1966) 237–238.