Symmetric Venn Diagrams and Symmetric Chain Decompositions

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A Venn diagram for n sets is symmetric if the n closed curves are symmetric rotations of the same curve. The existence of such diagrams for n sets is possible only for primes n, and they were initially constructed for primes $n \leq 7$. Hamburger devised an astonishing construction for $n = 11$. For his inspiration he credited the Greene-Kleitman bracketing construction of a symmetric chain decomposition (SCD) of the Boolean lattice $\mathcal{B}_n$ of all subsets of $[n] := \{1, \ldots, n\}$, ordered by inclusion.

In my study, it became apparent that one might be able to apply the bracketing construction to produce Venn diagrams for all primes n, a project which was successfully completed (G.-Killian-Savage 2004): The key ingredient of the proof is the construction of a SCD of the "Necklace Poset" N_n, n prime, in which each element consists of a subset of $[n]$ and its cyclic rotations. That is, N_n is the quotient poset $\mathcal{B}_n/\mathbb{Z}_n$, consisting of orbits of the Boolean lattice $\mathcal{B}_n$ under the action of the cyclic group $\mathbb{Z}_n$. Researchers were convinced that the Necklace Poset should actually have a SCD for all n, prime or composite, but the previous method worked only for primes n.

My student Kelly Kross Jordan devised an insightful new method to prove that N_n does indeed have a SCD for general n. Tantalizing challenges remain open both on SCD's for general quotient posets $\mathcal{B}_n/G$ and on constructing "simple" symmetric Venn diagrams, in which no three curves meet at a point. Mamakani and Ruskey can construct such diagrams for $n = 11$. Current undergrad Emily Theus is investigating the "Reversible Necklace Poset" $\mathcal{B}_n/D_{2n}$, where we instead take orbits under the action of the dihedral group.