GRADUATE TEST-SPRING 2012, ODE'S

Solve any 6 (six) problems

1. Let \( I_n \) be the \( n \times n \) identity matrix and \( J \) be the \( n \times n \) Jordan block

\[
J = \lambda I_n + H
\]

with \( \lambda \) a given real constant and \( H \) the \( n \times n \) matrix

\[
H := (h_{j,k}) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
& & \ddots & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
& & & \ddots \\
& & & & 0
\end{bmatrix}.
\]

Namely,

\[
H := (h_{j,k}), j, k = 1, \ldots, n, h_{j,k} = 1 \text{ if } k = j+1, j = 1, 2, \ldots, n-1, \ h_{j,k} = 0 \text{ if } k \neq j+1.
\]

a) Prove that \( M = \exp(tJ) \) is a fundamental matrix solution of the system

\[
\frac{d}{dt} M(t) = JM(t).
\]

b) Prove that

\[
\exp(tJ) = \exp(t\lambda) \left( I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \right).
\]

c) Determine the elements of the \( n \times n \) matrix

\[
\lim_{t \to \infty} t^{-(n-1)} \left( I_n + \sum_{k=1}^{n-1} \frac{t^k}{k!} H^k \right).
\]

2. A differential system is called complete if for any \( y_0 \in \mathbb{R}^n \) and any \( t_0 \in \mathbb{R} \) there exists a solution \( y(t) \) defined for all \( t \in \mathbb{R} \) and such that \( y(t_0) = y_0 \). Given the differential system with unknown \( y = (y_1, y_2) \in \mathbb{R}^2 \)

\[
\begin{align*}
\dot{y}_1 &= [\cos(t + y_1y_2)]^{23} + 3 \sin(y_1^5 + y_2^{2012}), \\
\dot{y}_2 &= \sin[\log(1 + y_1^2y_2^4)] + 3 \cos(y_1^5 + y_2^{2012}),
\end{align*}
\]

prove that it is complete.

3. Given the autonomous differential system

\[
\dot{y} = f(y), \text{ where } f \in C^1(\mathbb{R}^n),
\]

let \( y(t) \) be a \( n \times 1 \) column real-valued vector solution defined on \( \mathbb{R} \). Prove that if \( y(t_0 + P) = y(t_0) \) for some \( t_0 \in \mathbb{R} \) and some \( P > 0 \), then

\[
y(t + P) = y(t), \text{ for all } t \in \mathbb{R},
\]
i.e. \( y(t) \) is a periodic function of \( t \in \mathbb{R} \).

4. Given the autonomous differential system
\[
\dot{y} = f(y), \quad \text{where } f \in C^1(\mathbb{R}^n),
\]
let \( y(t) \) be an \( n \times 1 \) column real-valued vector solution defined on \( (a, b) \). Prove the following:

(i) If \( \lim_{t \to \infty} y(t) = L \) (\( L \) being a constant vector in \( \mathbb{R}^n \)), then \( L \) is a critical point for the system.

(ii) Assume \( \dot{y} \) is not identically zero and that \( \lim_{t \to b^+} y(t) = L \), where \( L \) is a critical point for the system. Then \( b = \infty \).

5. Let \( f(x, y) = x^2y + y^3 \) and \( g(x, y) = y(1 - x^2 - 2y^2) - 2x^3 - 2xy^2 \).
(a) Prove that the system below has periodic solutions
\[
\dot{x} = f(x, y), \quad \dot{y} = g(x, y).
\]
(b) Prove that the system below has non-periodic, non-constant solutions
\[
\dot{x} = -x + f(x, y), \quad \dot{y} = g(x, y).
\]

6. Consider the system
\[
\dot{x} = x + xy, \quad \dot{y} = y + xy.
\]
(a) Find the equilibria and study their stability.
(b) Prove that \( x = 0, y = 0 \) and \( x = y \) are the only invariant lines.
(c) Find the null and vertical (possibly curvilinear) clines.
(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. (a) Solve the initial value problem
\[
\dot{x} = x + y + z, \quad \dot{y} = y + z, \quad \dot{z} = -z \quad \text{with } x(0) = y(0) = z(0) = 1.
\]
(b) Find all initial vectors \( (x(0), y(0), z(0)) \) for which the solution converges to zero as \( t \) converges to infinity.
(c) Find all initial vectors \( (x(0), y(0), z(0)) \) for which the solution converges to zero as \( t \) converges to negative infinity.

8. Consider the equation
\[
\ddot{x} + \cos x = 0.
\]
(a) Find all the critical points and discuss their stability.
(b) Find the level set of the Hamiltonian which contains all the unstable critical points, i.e. find the constant \( c \in \mathbb{R} \) such that the planar curve of equation \( H(x, y) = c \) contains all the unstable critical points.
(c) Draw the phase diagram and use arrows to indicate the direction of increasing time (independent variable).
ENTRANCE EXAM: Differential Equations (August 30, 2012)

There are 8 (eight) problems on this test. Solve any 6 (six) of them. Clearly motivate your answers/solutions.

1. Let $M(t)$ be a square matrix and consider the matrix differential equation

$$\frac{dM(t)}{dt} = AM(t), \ A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}. $$

(a) Determine a fundamental solution.

(b) Determine all column vector solutions $y(t)$ to

$$\frac{dy(t)}{dt} = Ay(t)$$

that converge to the origin as $t \to \infty$.

(c) Determine all solutions that are bounded on $(-\infty, \infty)$.

2. A differential system is called complete if all of its real-valued solutions to all of its initial value problems exist for all $t \in \mathbb{R}$. Consider the differential system $(y_1, y_2 \in \mathbb{R})$

$$\begin{cases} \dot{y}_1 = 3 \cos(\exp(y_1 y_2)) + y_1 y_2 (1 + y_1^2 + y_2^2)^{-1}, \\ \dot{y}_2 = \cos(1 + y_1^2 + y_2^2) + \sin(y_1^5 + y_2^{20}). \end{cases} \quad (1)$$

Prove that (1) is complete.

3. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable vector field $f = f(y, p)$ with $y \in \mathbb{R}^n, p \in \mathbb{R}$. Let $y(t, p)$ be an $n$ by 1 vector valued solution of the IVP:

$$\frac{dy(t, p)}{dt} = f(y(t, p), p), \ t, p \in \mathbb{R}, \text{ with } y(t_0, p) = y_0 \in \mathbb{R}^n \text{ for all } p \in \mathbb{R}. \quad (2)$$

Prove that

$$\frac{\partial y}{\partial p}(t, p) = \int_{t_0}^{t} \frac{\partial f}{\partial p}(y(s, p), p)ds + \int_{t_0}^{t} J \frac{\partial y}{\partial p}(s, p)ds,$$

where $J$ is the Jacobian matrix

$$J = (a_{k,j}), \ a_{k,j} := \frac{\partial f_k}{\partial y_j}(y, p), \ k, j = 1, \ldots, n.$$

Hint: Convert (2) into an integral equation.
4. Let

\[ A(t) := \begin{bmatrix} t & \sin t \\ \cos t & -t \end{bmatrix}. \]

Denote by \( M(t) \) a 2 by 2 matrix solution of the linear matrix equation

\[ \frac{dM(t)}{dt} = A(t)M(t), \quad -\infty < t < \infty. \]

Prove that \( \det M(t) \), the determinant of \( M(t) \), is a constant on \((-\infty, \infty)\).

5. (a) Find and classify all the equilibria of the system

\[ \dot{x} = -6y + 2xy - 8, \quad \dot{y} = -x^2 + y^2. \]

(b) Sketch the phase diagram of the system from (a) (make sure to draw arrows to show direction of increasing time).

6. Consider the system

\[ \dot{x} = x - xy, \quad \dot{y} = y - xy. \]

(a) Find the equilibria and study their stability.

(b) Prove that \( x = 0, y = 0 \) and \( x = y \) are the only invariant lines.

(c) Find the null and vertical (possibly curvilinear) clines.

(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system

\[ \dot{x} = x - y - 2x^2, \quad \dot{y} = x + y - 2y^2. \]

(a) Discuss the stability of the origin.

(b) Determine if there is a periodic solution. In either case, show your proof.

8. Consider the initial value problem (IVP)

\[ \ddot{x} - 2x - 2x^3 = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = \beta. \]

(a) Use the Hamiltonian energy of the solution \( x(t) \) to show that if \( \alpha = 1, \beta = 0 \), then \( x(t) \geq 1 \) on its maximal interval of existence.

(b) Find the solution with \( \alpha = 0, \beta = 1 \). What is its maximal interval of existence? (Recall: the maximal interval of existence is the interval around \( t = 0 \) beyond which the solution is undefined.)
Graduate exam 2013, ODE’s

Do any 6 problems. All problems carry the same weight. Explain your solutions.

1. Solve the initial value problem

\[
\dot{y} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} y - \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y(7) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.
\]

Determine \( \lim_{t \to \infty} y(t) \).

2. Let \( A \) be a constant \( n \times n \) matrix with eigenvalues \( \lambda_j, j = 1, 2, \ldots, n \). Let \( \rho := \max\{\Re(\lambda_j) : j = 1, 2, \ldots, n\} \). Consider a fixed vector solution of \( \dot{y} = Ay(t) \) on the interval \([a, \infty)\). Prove that for every \( \epsilon > 0 \), there exists \( K > 0 \) such that

\[ |y(t)| \leq Ke^{(\rho + \epsilon)t} \text{ for all } t \in [a, \infty), \]

where \(|·|\) is a suitable norm.

3. Consider the autonomous differential system

\[
\dot{y} = f(y), \text{ where } f \in C^1(\mathbb{R}^n).
\]

Let \( y(t) \) be a continuous \( n \times 1 \) column vector solution of (1) on \((a, b)\). Prove the following:

(i) Assume \( \dot{y} \) is not identically zero and that \( \lim_{t \to b^+} y(t) = L \), where \( L \) is a critical point of (1). Then \( b = \infty \).

(ii) \( y(t + \omega) \) is also a vector solution of (1) on \((a + \omega, b + \omega)\).

4. (i) Calculate the Jacobian matrix of \( f \) if

\[
f(t, y) = A(t)y + g(t).
\]
(ii) Prove in detail (using the theorem of existence and uniqueness for general systems), that if $A(t)$ and $g(t)$ possess continuous entries on the closed interval $[c, d]$, then every initial value problem

$$\dot{y} = A(t)y + g(t), \quad y(t_0) = \eta, \quad t_0 \in [c, d], \quad \eta \in \mathbb{R}^n$$

possesses a unique solution on $[c, d]$.

5. Discuss the stability of the zero solution for

$$\dot{x} = -x \cos y + xy + ye^{-t}, \quad \dot{y} = \sin x - 2y + 2x \sin y + x \sin \frac{1}{1 + t^2}.$$

6. (i) (2 points) Show that the origin is the only equilibrium solution for

$$\dot{x} = -x^3 - xy^4, \quad \dot{y} = -y^3 - x^2 y.$$

(ii) Show that the equilibrium is globally asymptotically stable (which means that every solution, no matter where it starts, approaches this equilibrium forward in time).

(iii) Does this system have nonconstant periodic solutions?

7. Consider the scalar equation $\ddot{x} + 2x^3 = 0$.

(i) Sketch the phase diagram, find the critical points and study their stability.

(ii) Assume $x(0) = 1$, $\dot{x}(0) = \sqrt{3}$. Prove that the time it takes the corresponding orbit to cross the $x$-axis in phase plane is $t_0 = \int_0^{\sqrt{2}} \frac{ds}{\sqrt{4 - s^4}}$.

(iii) Show that $t_0$ defined at $(c)$ is finite (note that the integral above is improper, so it is necessary to prove that it is convergent).

8. Consider the system

$$\dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2 y - 4y^3.$$

(i) Find all critical points and discuss their stability. Hint: First use the Lyapunov function $V(x, y) = 2x^2 + y^2$ to prove that if there are other critical points beside the origin, then they must lie on the unit circle.

(ii) Are there any periodic solutions? If so, approximately where are their orbits located?
**ENTRANCE EXAM:** Differential Equations, Fall 2013.
Solve any 6 (six) problems. All problems carry the same weight.

1. Given the initial value problem

   \[ y' = f(t, y, p), \quad y(t_0) = \eta, \]

   \[ t, t_0, p \in \mathbb{R}^1, \quad y, \eta, f \in \mathbb{R}^n. \]

   Assume that \( f(t, y, p) \) is a continuous vector function and that the entries of the Jacobian matrix

   \[ J_f := \left( \frac{\partial f_i}{\partial y_j} \right)_{ij} = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_n \end{bmatrix} \]

   are continuous in the set of points \((t, y, p)\), where

   \[ t, t_0, p \in \mathbb{R}^1, \quad y, \eta, f \in \mathbb{R}^n. \]

   Consider the solution \( y = \phi(t, t_0, \eta, p) \) as a vector function of the variables \((t, t_0, \eta, p)\). Determine an equation that is satisfied by a) \( \frac{\partial \phi}{\partial t_0} \), b) \( \frac{\partial \phi}{\partial p} \). What information could be derived from these equations? Why is that information important?

2. Convert the scalar ode \( x'' + x^2 = 0 \) into a vector system of differential equations. Show, that a solution to any initial value problem of the ode exists on any interval \([a, b]\).

3. a) Explain the essence of the method of successive approximations and under what assumptions it is valid. b) Calculate the first 3 successive approximations for the initial value problem

   \[ y' = \begin{bmatrix} -y_1 + y_2^2 \\ y_1 + y_2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

4. Provide the best estimates for the maximal intervals of existence of solutions of

   \[ y' = \begin{bmatrix} (t + 5)^{-2} & t \\ t^2 & (t - 1)^{-3} \end{bmatrix} y, \quad y(t_0) = \eta, \]

   with a) \( t_0 = -10 \), b) \( t_0 = 0 \), c) \( t_0 = 2 \).
5. (a) Find and classify all the equilibria of the system
\[ \dot{x} = -3y + xy - 4, \quad \dot{y} = x^2 - xy. \]
(b) Sketch the phase diagram.

6. Consider the system
\[ \dot{x} = x - xy, \quad \dot{y} = y + xy. \]
(a) Find the equilibria and study their stability.
(b) Find all the invariant lines.
(c) Find the null and vertical (possibly curvilinear) clines.
(d) Draw the phase diagram and use arrows to indicate the direction of increasing time.

7. Consider the system
\[ \dot{x} = 2x + y - 2x^3 - 3xy^2, \quad \dot{y} = -2x + 4y - 2x^2y - 4y^3. \]
(a) (6 points) Find all critical points and discuss their stability. *Hint: First use the Lyapunov function \( V(x, y) = 2x^2 + y^2 \) to prove that if there are other critical points beside the origin, then they must lie on the unit circle.*
(b) (4 points) Are there any periodic solutions? If so, approximately where are their orbits located?

8. Consider the linear system \( \dot{x} = Ax \), where \( A \) is an \( n \times n \) nonsingular, antisymmetric (i.e. \( A^T = -A \)), real matrix.
(a) Show that every solution satisfies \( |x(t)| = \text{const} \) for all \( t \in \mathbb{R} \).
(b) Find all critical points and study their stability.
(c) If \( n = 2 \), are there any non-periodic solutions?
1. Let $y^T = (y_1, y_2, ..., y_n)$ denote a row vector that is the transpose of $y \in \mathbb{R}^n$. In particular let $0^T = (0, ..., 0)$ be the transpose of the zero vector. Let $f^T(y, t) := (f_1(y, t), f_2(y, t), ..., f_n(y, t))$ be a vector field in $\mathbb{R}^n$ where $f_k(y, t) \in \mathbb{R}$, $k = 1, 2, ..., n$ and $t \in [a, b]$. Formulate (without proof) an existence and uniqueness theorem that will ensure that the initial value problem

$$\frac{dy}{dt}(t) = f(y, t), \; y(t_0) = \eta, \; t_0 \in [a, b],$$

possesses a unique solution on some subinterval of $[a, b]$. Based on this formulation prove that if

$$|f(y, t)| \leq A|y| + B, \quad y \in \mathbb{R}^n, \; t \in [a, b],$$

with some nonnegative constants $A$ and $B$, then the initial value problem (1) possesses a unique solution on $[t_0, b]$.

2. (a) Determine the value of all constants $c$ that will guarantee the existence of a bounded non trivial (not zero) vector solution $y(t)$ on $(-\infty, \infty)$ to the linear system with constant coefficients

$$y' = \begin{bmatrix} 1 & c \\ 3 & 2 \end{bmatrix} y.$$

(b) Let $A = A^T$ be a constant symmetric $n \times n$ matrix with real entries. Prove that the system

$$\frac{dy}{dt}(t) = Ay(t),$$

possesses a bounded non trivial (non zero) vector solution $y(t)$ on $(-\infty, \infty)$ iff the matrix $A$ has a zero eigenvalue. *Hint: Use properties of symmetric matrices.*

3. Prove that the equation below has periodic solutions

$$\ddot{x} + (\dot{x})^5 + x^2 \dot{x} - \dot{x} + x^3 = 0.$$
4. Solve the initial value problem

\[
\dot{y} = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix} y - \begin{bmatrix}
4 \\
4 \\
10
\end{bmatrix}, \quad y(6) = \begin{bmatrix}
0 \\
4 \\
0
\end{bmatrix}.
\]

5. Provide examples for the following:
   (a) An initial value problem \( \dot{y} = f(y), \; y(0) = 0 \) with \( f(y) \in C(\mathbb{R}) \) such that the initial value problem does not have a unique solution.
   (b) An initial value problem \( \dot{y} = f(y), \; y(0) = y_0 \) with \( f(y) \in C(\mathbb{R}) \), with solutions that do not exist on \( \mathbb{R} \).
   (c) A scalar singular differential equation such that non-trivial solutions exist on \( \mathbb{R} \).

6. Consider the system

\[
\dot{x} = y + x^2 y, \quad \dot{y} = -4x + x^3 - xy^2.
\]

   (a) Show that the system is Hamiltonian and find its Hamiltonian function.
   (b) Find the critical points and discuss their stability.
   (c) Sketch the phase diagram (make sure to put arrows to indicate the direction of motion along orbits).

7. Consider the differential system

\[
\dot{x} = x^3 + x^2 y, \quad \dot{y} = y^3 + y^2 z, \quad \dot{z} = z^3 + z^2 x.
\]

   (a) Show that all solutions exist for all negative times;
   (b) Show that all solutions converge to the origin as \( t \to -\infty \);
   (c) Show that with the exception of the stationary solutions (find them!), all other solutions blow up in finite time;
   (d) Study the stability of the critical points.

8. (a) Discuss the stability of the null solution for the system:

\[
\dot{x} = x - e^{-t} y, \quad \dot{y} = y - z(1 + e^{-t}), \quad \dot{z} = 2z + xyz^2.
\]

   (b) Is the origin asymptotically stable for the system:

\[
\dot{x} = x^2 - y^2 z, \quad \dot{y} = x y^3 - z, \quad \dot{z} = -2xy?
\]

   Hint for (b): Study the evolution of a point in a certain octant.